Constructive Comfort-compactness

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17.11.2016

Comfort 1968

Comfort used the notion of maximal ideal to **define** classical topological compactness.

He wanted to avoid the use of the axiom choice in the construction of the Stone-Čech compactification and the proof of the Tychonoff theorem.

> *It is my feeling, however, that the definition of compactness relative to which the theorems of Stone-Čech and Tychonoff are unprovable without the axiom of choice is, from the point of view of topological analysis and the theory of rings of continuous functions,* **unnatural** *and* **unsuitable***.*

What Comfort did

1. He defined a topological space to be compact, if it is a completely regular Hausdorff space for which each maximal ideal in $C^*(X)$ is fixed.

2. Using classical logic, but avoiding the axiom of choice, he showed many expected properties for his notion of compactness.

3. Based on a theorem of Stone-Čech type he proved the corresponding Tychonoff theorem.

For this proof of Tychonoff theorem he writes:

> *My own attempts to prove this result "directly" have been unsuccessful, and this lends some interest to the proof of (the Tychonoff) Theorem ... and the theorem of Stone-Čech type upon which it depends.*

Bishop spaces

A Bishop space (function space) is a function-theoretic constructive alternative to the notion of topological space.

Bishop 1967, Bridges 2012, Ishihara 2013, P. 2015

It is a theory within BISH^*.

Formal counterpart to BISH^*: Myhill's CST^*, or $\mathrm{CZF} + \mathrm{REA} + \mathrm{DC}$.

Continuity as a primitive notion

A **Bishop space** is a pair $\mathcal{F} = (X, F)$, where X is an inhabited set and $F \subseteq \mathbb{F}(X)$, a **Bishop topology**, or simply a **topology**, satisfies the following conditions:

(BS_1) $a \in \mathbb{R} \to \overline{a} \in F$.
(BS_2) $f \in F \to g \in F \to f + g \in F$.
(BS_3) $f \in F \to \phi \in \mathrm{B}(\mathbb{R}) \to \phi \circ f \in F$,
(BS_4) $f \in \mathbb{F}(X) \to U(F, f) \to f \in F$,

If $f, g \in \mathbb{F}(X)$, $\epsilon > 0$, and $\Phi \subseteq \mathbb{F}(X)$, we define $U(g, f, \epsilon)$ and $U(\Phi, f)$ by

$$U(g, f, \epsilon) :\Leftrightarrow \forall_{x \in X}(|g(x) - f(x)| \leq \epsilon),$$

$$U(\Phi, f) :\Leftrightarrow \forall_{\epsilon > 0}\exists_{g \in \Phi}(U(g, f, \epsilon)).$$

$$fg, \lambda f, -f, f \vee g, f \wedge g, |f| \in F$$

$$\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$$

A **morphism** from $\mathcal{F} = (X, F)$ to $\mathcal{G} = (Y, G)$ is a function $h : X \to Y$ such that

$$\forall_{g \in G}(g \circ h \in F).$$

It captures uniform continuity!
We denote $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ the set of the morphisms from $\mathcal{F}$ to $\mathcal{G}$.
$F = \mathrm{Mor}(\mathcal{F}, \mathcal{R})$, where $\mathcal{R} = (\mathbb{R}, \mathrm{B}(\mathbb{R}))$ is the **Bishop space of reals**.

The least topology $\bigvee F_0$ generated by a given **subbase** $F_0 \subseteq \mathbb{F}(X)$

$$\frac{f_0 \in F_0}{f_0 \in \bigvee F_0} \quad \frac{a \in \mathbb{R}}{\overline{a} \in \bigvee F_0} \quad \frac{f, g \in \bigvee}{f + g \in \bigvee F_0},$$

$$\frac{f \in \bigvee F_0, \ \phi \in \mathrm{B}(\mathbb{R})}{\phi \circ f \in \bigvee F_0} \quad \frac{(g \in \bigvee F_0, \ U(g, f, \epsilon))_{\epsilon > 0}}{f \in \bigvee F_0},$$

$$\frac{g_1 \in \bigvee F_0 \ \wedge \ U(g_1, f, \frac{1}{2}), \ g_2 \in \bigvee F_0 \ \wedge \ U(g_2, f, \frac{1}{2^2}), \ g_3 \in \bigvee F_0 \ \wedge \ U(g_3, f, \frac{1}{2^3}), \ldots}{f \in \bigvee F_0}$$

$$\begin{aligned}
&\forall_{f_0 \in F_0}(P(f_0)) \rightarrow \\
&\forall_{a \in \mathbb{R}}(P(\overline{a})) \rightarrow \\
&\forall_{f, g \in \bigvee F_0}(P(f) \rightarrow P(g) \rightarrow P(f + g)) \rightarrow \\
&\forall_{f \in \bigvee F_0} \forall_{\phi \in \mathrm{B}(\mathbb{R})}(P(f) \rightarrow P(\phi \circ f)) \rightarrow \\
&\forall_{f \in \bigvee F_0}(\forall_{\epsilon > 0} \exists_{g \in \bigvee F_0}(P(g) \ \wedge \ U(g, f, \epsilon)) \rightarrow P(f)) \rightarrow \\
&\forall_{f \in \bigvee F_0}(P(f)).
\end{aligned}$$

Lifting of morphisms: If $\mathcal{G} = (Y, \mathcal{F}(G_0))$, then $h : X \rightarrow Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ if and only if $\forall_{g_0 \in G_0}(g_0 \circ h \in F)$.

A **base** Φ_0 of a topology F on X is an inhabited subset of F such that $\mathcal{U}(\Phi_0) = \{f \in \mathbb{F}(X) \mid U(\Phi_0, f)\} = F$.

Definition

Let $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ be Bishop spaces, $A \subseteq X$ is inhabited, and $\phi : X \to Y$ is onto Y. The **product** Bishop space $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$ of $\mathcal{F}$ and $\mathcal{G}$, **relative** Bishop space $\mathcal{F}_{|A} = (A, F_{|A})$ on A, and the **quotient topology** G_ϕ on Y are defined, respectively, by

$$F \times G := \bigvee [\{f \circ \pi_1, \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\}] =: \bigvee_{f \in F}^{g \in G} f \circ \pi_1, g \circ \pi_2,$$

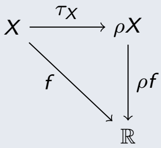
$$F_{|A} = \bigvee \{f_{|A} \mid f \in F\} =: \bigvee_{f \in F} f_{|A}.$$

$$F_\phi := \{g \in \mathbb{F}(Y) \mid g \circ \phi \in F\}.$$

F is **separating**, if $\forall_{x,y \in X}(\forall_{f \in F}(f(x) = f(y)) \rightarrow x = y)$.

Theorem (Stone-Čech theorem for Bishop spaces)

If $\mathcal{F} = (X, F)$ is a Bishop space, there exists a Bishop space $\rho\mathcal{F} = (\rho X, \rho F)$ and a mapping $\tau_X : X \rightarrow \rho X \in \mathrm{Mor}(\mathcal{F}, \rho\mathcal{F})$ such that:
(i) The topology ρF is separating.
(ii) The induced mapping $T_X : \rho F \rightarrow F$ of τ_X is an algebra and lattice isomorphism.
(iii) For every $f \in F$ there exists a unique $\rho f \in \rho F$ such that the following diagram commutes



Theorem (Tychonoff embedding theorem for Bishop spaces)

If $\mathcal{F} = (X, F)$ is a Bishop space, F is separating if and only if $\mathcal{F}$ is topologically embedded into the Euclidean Bishop space $\mathcal{R}^F$.

Theorem (The first base theorem)

If $F_0 \subseteq \mathbb{F}^*(X)$, then then $\bigvee_0 F_0$ is a base of $\bigvee F_0$.

If $\Phi : \mathrm{On} \to V$ is defined by

$$\Phi_0 = F_0,$$

$$\Phi_{\alpha+1} = \overline{\bigvee_0 \Phi_\alpha},$$

$$\Phi_\lambda = \bigcup_{\alpha<\lambda} \Phi_\alpha, \quad \lambda \text{ is a limit ordinal.}$$

then, classically,

$$\bigvee F_0 = \Phi_{\omega_1},$$

where ω_1 is the first uncountable limit ordinal.

Theorem (The second base theorem)

If $\mathcal{F} = (X, \bigvee F_0)$ is a Bishop space such that every element of F_0 is bounded and $\Phi \subseteq F$ such that

(i) $F_0 \subseteq \Phi$,
(ii) $\mathrm{Const}(X) \subseteq \Phi$,
(iii) Φ *is closed under addition and multiplication,*

then Φ *is a base for* $\bigvee F_0$.

Proof.

By the first base theorem one shows $\overline{\Phi} = \overline{\bigvee_0 F_0} = F$ (use of the Weierstrass approximation theorem). □

$$F \oplus G := \{\sum_{i=1}^{n}(f_i \circ \pi_X)(g_i \circ \pi_Y) \mid n \in \mathbb{N}, f_i \in F, g_i \in G, 1 \leq i \leq n\}.$$

$$\bigoplus_{n\in\mathbb{N}} F_n = \{\sum_{j=1}^{m} \phi_j \mid m \in \mathbb{N}, \phi_j \in \Sigma_0, 1 \leq j \leq m\},$$

$$\Sigma_0 := \{\prod_{k=1}^{n}(f_k \circ \pi_k) \mid n \in \mathbb{N}, f_k \in F_k, 1 \leq k \leq n\}.$$

Corollary

If (X, F) and (Y, G) are pseudo-compact Bishop spaces, $F \oplus G$ is a base for $F \times G$.

Corollary

If $\mathcal{F}_n = (X_n, F_n)$ is a sequence of pseudo-compact Bishop spaces and $\mathcal{F} = (X, F)$, where $X = \prod_{n\in\mathbb{N}} X_n$ and $F = \prod_{n\in\mathbb{N}} F_n$, then $\bigoplus_{n\in\mathbb{N}} F_n$ is a base for F.

$$M(f) := \{|f(x)| \mid x \in X\},$$
$$N(f) := \{a \geq 0 \mid |f| \leq \overline{a}\}.$$

Definition

If F is a topology on X and $f \in F^*$, we call f **normable**, if $\sup M(f)$ exists and its norm is defined by
$$||f|| = \sup M(f).$$
We call f **weakly normable**, if $\mathrm{lub} M(f)$ exists and its weak norm is defined by
$$||f||_w = \mathrm{lub} M(f).$$

If f is normable, then f is weakly normable and $||f||_w = ||f||$. Moreover, $||f||_w = \inf N(f)$.

Definition

If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, a function $T : F \to G$ is called a **ring homomorphism**, or simply a **homomorphism**, if

$$T(f_1 + f_2) = T(f_1) + T(f_2),$$

$$T(f_1 f_2) = T(f_1) T(f_2),$$

for every $f_1, f_2 \in F$. We denote the set of homomorphisms between F and G by $\mathrm{Hom}(F, G)$. A homomorphism T is called **non-zero**, if $T(\overline{1})(y) > 0$, for some $y \in Y$. We denote the set of non-zero homomorphisms between F and G by $\mathrm{Hom}^*(F, G)$. If $\tau : Y \to X$ is in $\mathrm{Mor}(\mathcal{G}, \mathcal{F})$, the **induced homomorphism** $T : F \to G$ from τ is defined by $T(f) := f \circ \tau$, for every $f \in F$.

$T(\overline{0}) = \overline{0}$ and $T(-f) = -T(f)$.
If $f \geq \overline{0}$, then $T(f) \geq \overline{0}$.
$T(|f|) = |T(f)|$.
$T(f_1 \vee f_2) = T(f_1) \vee T(f_2)$ and $T(f_1 \wedge f_2) = T(f_1) \wedge T(f_2)$.
$T(\overline{1})(y) \in 2$.
$T(\overline{n}) \leq \overline{n}$, for every $n \in \mathbb{N}$.
$T(F^*) \subseteq G^*$.
$T(\overline{a}f) = \overline{a}T(f)$ (**non-trivial**).
$|T(\overline{a}) \leq |\overline{a}|$.
If T is non-zero, then $T(\overline{a})$ is normable and $||T(\overline{a})|| = |a|$.
If $a \geq 0$, then $T(\overline{a}) \leq \overline{a}$.
If $T(f) \leq \overline{a}$, then $T(f) \leq T(\overline{a})$.
If $T(f) \geq \overline{a}$, then $T(f) \geq T(\overline{a})$.

Definition

If $G = \mathrm{Const}(\{x\}) \cong \mathbb{R}$, a homomorphism between F and $\mathbb{R}$ is called a **character** of F. We denote their set by $\mathrm{Char}(F)$ and the set of the non-zero characters of F by $\mathrm{Char}^*(F)$.

A non-zero character π of F is called **fixed**, if there exists $x \in X$ such that $\pi = \pi_x$, where

$$\pi_x(f) = f(x),$$

for every $f \in F$. In this case x is a **fixing point** for π, or x **fixes** π. We denote the set of fixed characters of F by $\mathrm{Char}^{**}(F)$.
The **kernel** $\ker(\pi)$ of a non-zero character π of F is defined by

$$\ker(\pi) := \{f \in F \mid \pi(f) = 0\}.$$

Clearly, if F separates the points of X, there is a unique fixing point for a fixed character of F.

Corollary

Let $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ be Bishop spaces.
(i) If G is 2-connected, and $T \in \mathrm{Hom}^*(F, G)$, then $T(\overline{1}) = \overline{1}$.
(ii) If $\pi \in \mathrm{Char}^*(F)$, then π is onto $\mathbb{R}$; in fact $\pi(\overline{a}) = a$, for every $a \in \mathbb{R}$.
(iii) If $\pi : \mathbb{R} \to \mathbb{R}$ is a non-zero ring homomorphism i.e., $\pi(1) > 0$, then π is the identity.

Proof.

(i) One shows that if G is 2-connected, $T(\overline{1})$ is constant, and since $T(\overline{1})(y) = 1$ for some $y \in Y$, $T(\overline{1}) = \overline{1}$.
(ii) Since $\mathbb{R}$ is isomorphic to $\mathrm{Const}(\{y\})$, which is a 2-connected topology, by (i) we get that $\pi(\overline{1}) = \overline{1}$ and $\pi(\overline{a}) = \overline{a}\pi(\overline{1}) = \overline{a}\overline{1} = \overline{a}$, which corresponds to a.
(iii) We consider $F = \mathrm{Const}(\{x\})$ and we use (ii). □

Proposition

If X is an inhabited set, $x \in X$, $F_0 \subseteq \mathbb{F}^*(X)$, and $\pi \in \mathrm{Char}^*(\bigvee F_0)$ such that $\pi(f_0) = f_0(x)$, for every $f_0 \in F_0$, then $\pi = \pi_x$.

Proposition

Let $T \in \mathrm{Hom}(F, G)$ *be an injection,* $\pi \in \mathrm{Char}^*(F^*)$*, and* $f \in F^*$*.*
(i) If $f, T(f)$ *are weakly normable, then* $||T(f)||_w = ||f||_w$*.*
(ii) If $f, T(f)$ *are normable, then* $||T(f)|| = ||f||$*.*
(iii) If f *is normable, then* $\sup\{|\pi(f)| \mid \pi \in \mathrm{Char}^*(F^*)\}$ *exists and*

$$||f|| = \sup\{|\pi(f)| \mid \pi \in \mathrm{Char}^*(F^*)\}.$$

Theorem (The character extension-theorem (CET))

Let $(X, F), (Y, G)$ be Bishop spaces, $F = F^*$, and Φ a base for F which includes $\mathrm{Const}(X)$ and it is closed under addition and multiplication. If $\pi : \Phi \to G$ is a ring homomorphism, there exists a unique homomorphism $\Pi : F \to G$ which extends π.

Corollary

Let (X, F) be a Bishop space such that $F = F^*$, and Φ a base for F which includes $\mathrm{Const}(X)$ and it is closed under addition and multiplication. If $\pi : \Phi \to \mathbb{R}$ is a ring homomorphism, there exists a unique character $\Pi : F \to \mathbb{R}$ of F which extends π. Moreover, if there exists some $x \in X$ such that $\pi(\theta) = \theta(x)$, for every $\theta \in \Phi$, then $\Pi = \pi_x$.

Proof.

If Y is an inhabited set, then $\mathrm{Const}(X)$ is ring-isomorphic to $\mathbb{R}$, and we use the CET. Since π_x also extends π, by the uniqueness of the character extension we get that $\Pi = \pi_x$. □

Corollary

Let $(X, \bigvee F_0)$ be a Bishop space such that $F_0 \subseteq F^*(X)$, and $\Phi \subseteq \bigvee F_0$ such that Φ includes the sets $\mathrm{Const}(X)$ and F_0, and it is closed under addition and multiplication. If (Y, G) is a Bishop space and $\pi : \Phi \to G$ is a ring homomorphism, there exists a unique homomorphism $\Pi : \bigvee F_0 \to G$ which extends π.

$$F \oplus G := \{\sum_{i=1}^{n}(f_i \circ \pi_X)(g_i \circ \pi_Y) \mid n \in \mathbb{N}, f_i \in F, g_i \in G, 1 \leq i \leq n\}.$$

Proposition

Let $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ *be Bishop spaces,* $\pi \in \mathrm{Char}^*(F)$*, and* $\varpi \in \mathrm{Char}^*(G)$*. The function* $\pi \oplus \varpi : F \oplus G \to \mathbb{R}$ *defined by*

$$(\pi \oplus \varpi)(\sum_{i=1}^{n}(f_i \circ \pi_X)(g_i \circ \pi_Y)) := \sum_{i=1}^{n} \pi(f_i)\varpi(g_i),$$

is a non-zero ring homomorphism.

Corollary

(i) There exists a unique element of $\mathrm{Char}^*(F \times G)$ *which extends* $\pi \oplus \varpi$.
(ii) If $\pi = \pi_x$*, for some* $x \in X$*, and* $\varpi = \varpi_y$*, for some* $y \in Y$*, their unique character extension on* $F \times G$ *is fixed and* (x, y) *is a fixing point for it.*

Proof.

(i) Since $F \oplus G$ is a base for $F \times G$, and since $\pi \oplus \varpi : F \oplus G \to \mathbb{R}$ is a ring homomorphism, there is a unique character extension of $\pi \oplus \varpi$ on $F \times G$.
(ii) If $\Pi_{(x,y)}$ is the fixed character on $F \times G$ fixed by (x, y), then

$$\begin{aligned}
(\pi_x \oplus \varpi_y)(\sum_{i=1}^{n}(f_i \circ \pi_X)(g_i \circ \pi_Y)) &= \sum_{i=1}^{n} \pi_x(f_i)\varpi_y(g_i) \\
&= \sum_{i=1}^{n} f_i(x)g_i(y) \\
&= \sum_{i=1}^{n}(f_i \circ \pi_X)(x, y)(g_i \circ \pi_Y)(x, y) \\
&= [\sum_{i=1}^{n}(f_i \circ \pi_X)(g_i \circ \pi_Y)](x, y) \\
&= \Pi_{(x,y)}(\sum_{i=1}^{n}(f_i \circ \pi_X)(g_i \circ \pi_Y)).
\end{aligned}$$

Proposition

Suppose that $(X, F), (Y, G)$ *are Bishop spaces,* $\Pi \in \mathrm{Char}(F \times G)$*, and that* $\pi_\Pi : F \to \mathbb{R}$ *and* $\varpi_\Pi : G \to \mathbb{R}$ *are defined, for every* $f \in F$ *and* $g \in G$*, respectively, by*

$$\pi_\Pi(f) = \Pi(f \circ \pi_X), \quad \varpi_\Pi(g) = \Pi(g \circ \pi_Y).$$

(i) $\pi_\Pi \in \mathrm{Char}(F)$ *and* $\varpi_\Pi \in \mathrm{Char}(G)$*.*

(ii) If $\Pi \in \mathrm{Char}^*(F \times G)$*, then* $\pi_\Pi \in \mathrm{Char}^*(F)$ *and* $\varpi_\Pi \in \mathrm{Char}^*(G)$*.*

(iii) If $\Pi \in \mathrm{Char}^{**}(F \times G)$ *and* $(x, y) \in X \times Y$ *fixes* Π*, then* $\pi_\Pi \in \mathrm{Char}^{**}(F)$*,* $\varpi_\Pi \in \mathrm{Char}^{**}(G)$ *such that* x *fixes* π_Π *and* y *fixes* ϖ_Π*.*

(iv) If F*,* G *are pseudo-compact and* $\Pi \in \mathrm{Char}^*(F \times G)$*, then* $\Pi = \pi_\Pi \oplus \varpi_\Pi$*, and if* $\pi \in \mathrm{Char}^*(F)$*,* $\varpi \in \mathrm{Char}^*(G)$ *such that* $\Pi = \pi \oplus \varpi$*, then* $\pi = \pi_\Pi$ *and* $\varpi = \varpi_\Pi$*.*

All results on the characters of the finite product of Bishop topologies extend to the case of the countable product of pseudo-compact Bishop topologies.

$D \subseteq X$ is F-**dense** in X, if for all $f, g \in F$, such that $f_{|D} = g_{|D}$, then $f = g$.
$A \subseteq X$ is inhabited, we $\subseteq X$ inhabited is **embedded** in X, if $\forall_{g \in F_{|A}} \exists_{f \in F}(g = f_{|A})$ i.e., if $F_{|A} = \{f_{|A} \mid f \in F\}$.

Proposition

If $\mathcal{F} = (X, F)$ is a pseudo-compact space and $A \subseteq X$ is inhabited, the following are equivalent.
(i) A is F-dense and embedded in X.
(ii) For every $\pi \in \mathrm{Char}^*(F)$, the mapping $\pi_{|A} : F_{|A} \to \mathbb{R}$, defined by $\pi_{|A}(f_{|A}) = \pi(f)$, for every $f \in F$, is in $\mathrm{Char}^*(F_{|A})$.

Proposition

Let $\mathcal{F} = (X, F)$ be a pseudo-compact space and $A \subseteq X$ inhabited. If $\varpi \in \mathrm{Char}^*(F_{|A})$, the map $\varpi^* : F \to \mathbb{R}$, where $\varpi^*(f) = \varpi(f_{|A})$, for every $f \in F$, is in $\mathrm{Char}^*(F)$. Moreover, if $a \in A$ fixes ϖ, then a fixes ϖ^*.

Proposition

Let $\mathcal{F} = (X, F)$ be a pseudo-compact space and $A \subseteq X$ inhabited. If $\varpi_1, \varpi_2 \in \mathrm{Char}^*(F_{|A})$ such that $\varpi_1(f_{|A}) = \varpi_2(f_{|A})$, for every $f \in F$, then $\varpi_1 = \varpi_2$.

$$U(f) := \{x \in X \mid f(x) > 0\}$$

$$A \text{ is } F\text{-closed } \leftrightarrow \forall_{x \in X}(\forall_{f \in F}(f(x) > 0 \rightarrow \exists_{a \in A}(f(a) > 0) \rightarrow x \in A),$$

$$\overline{A} := \{x \in X \mid \forall_{f \in F}(f(x) > 0 \rightarrow \exists_{a \in A}(f(a) > 0)\}.$$

Definition

Let F be a pseudo-compact topology on some X that separates the points of X, and $A \subseteq X$ inhabited. The notions A is *c*-**closed** and the *c*-**closure** of A are defined respectively by

$$A \text{ is } c\text{-closed } \leftrightarrow \forall_{x \in X}(\varpi_{x|A} \in \mathrm{Char}_0^*(F_{|A}) \rightarrow x \in A),$$

$$c(A) := \{x \in X \mid \varpi_{x|A} \in \mathrm{Char}_0^*(F_{|A})\},$$

where

$$\mathrm{Char}_0^*(F_{|A}) := \{\varpi_{|F_0(A)} \mid \varpi \in \mathrm{Char}^*(F_{|A})\},$$

$$F_0(A) := \{f_{|A} \mid f \in F\},$$

and $\varpi_{x|A} : F_0(A) \rightarrow \mathbb{R}$ is defined by

$$\varpi_{x|A}(f_{|A}) = \varpi_x(f) = f(x),$$

for every $f \in F$.

Proposition

Let F be a pseudo-compact topology on X that separates its points and $A, B \subseteq X$ inhabited.
(i) $A \subseteq c(A)$.
(ii) A is c-closed if and only if $c(A) = A$.
(iii) If $A, B \subseteq X$ are c-closed and $A \cap B$ is inhabited, then $A \cap B$ is c-closed.
(iv) $A \subseteq B \rightarrow c(A) \subseteq c(B)$.

Theorem

If F is a pseudo-compact topology on X that separates its points and $A \subseteq X$ is inhabited, then A is embedded in $c(A)$ and A is $F_{|c(A)}$-dense in $c(A)$.

Corollary

If F is a pseudo-compact topology on X that separates its points and $A \subseteq X$ is inhabited, then $c(A)$ is the least c-closed set including A.

Corollary (CLASS)

Let F be a pseudo-compact topology on X that separates its points and $A \subseteq X$ inhabited.
(i) $c(A) \subseteq \overline{A}$.
(ii) If A is F-closed, then A is c-closed.

Corollary

$c(0,1) = [0,1]$.

Proof.

It is immediate that $(0,1)$ is $C_u([0,1])$-dense in $[0,1]$ and embedded in $[0,1]$, hence $\pi_{1|(0,1)} \in \mathrm{Char}^*(\mathrm{B}(0,1))$. Hence $\pi_{1|(0,1)}(f_{|(0,1)}) = \pi_1(f) = f(1)$, for every $f \in C_u([0,1])$ i.e., $1 \in c(0,1)$. Similarly we show that $0 \in c(0,1)$. □

A deficiency of Comfort-compactness is that one needs the axiom of choice to show that such a Comfort-compact space is pseudo-compact (Feferman 1965: it is consistent with ZFC that every ultrafilter on $\mathbb{N}$ is fixed, and $\mathbb{N}$ is not pseudo-compact.).

There is a bijection between maximal ideals of $C^*(X)$ and its characters.

We suppose that our space is already pseudo-compact.

Definition

If F is a Bishop topology on some inhabited set X that separates the points of X, we call F *c*-**compact**, or **Comfort-compact**, if it pseudo-compact and

$$\forall_{\pi \in \mathrm{Char}^*(F)} \exists_{x \in X} (\pi = \pi_x).$$

Proposition

If $\mathcal{F} = (X, F)$ is a c-compact Bishop space and $A \subseteq X$ is inhabited, then $F_{|A}$ is c-compact if and only if A is c-closed.

Proof.

Suppose that A is c-compact, $x \in X$ and $\varpi \in \mathrm{Char}^*(F_{|A})$ such that $\forall_{f \in F}(\varpi(f_{|A}) = f(x))$. If $a \in A$ is a fixing point for ϖ, then $\forall_{f \in F}(\varpi(f_{|A}) = f_{|A}(a) = f(a))$, therefore $\forall_{f \in F}(f(a) = f(x))$. Since F separates the points of X, we have that $a = x$, hence $x \in A$. If A is c-closed, then if $\varpi \in \mathrm{Char}^*(F_{|A})$, the map $\pi : F \to \mathbb{R}$, where $\pi(f) = \varpi(f_{|A})$, for every $f \in F$, is in $\mathrm{Char}^*(F)$. Since F is c-compact, there exists some $x \in X$ that fixes π, therefore $\forall_{f \in F}(\pi(f) = \varpi(f_{|A}) = f(x))$. Since A is c-closed, $x \in A$, hence $\forall_{f \in F}(\varpi(f_{|A}) = f_{|A}(x))$. Since $x \in A$ fixes ϖ on every element of the subbase of $F_{|A}$, x fixes ϖ. □

Theorem (countable Tychonoff theorem for c-compact Bishop spaces)

If $\mathcal{F}_n = (X_n, F_n)$ is a c-compact Bishop space, for every $n \in \mathbb{N}$, the product $\prod_{n\in\mathbb{N}} \mathcal{F}_n = (\prod_{n\in\mathbb{N}} X_n, \prod_{n\in\mathbb{N}} F_n)$ is c-compact.

Proof.

The countable product of pseudo-compact topologies that separate the points of the corresponding spaces is pseudo-compact and separates the points of the product space. If $\varpi_n \in \mathrm{Char}^*(F_n)$, for every $n \in \mathbb{N}$, the function $\bigoplus_{n\in\mathbb{N}} \varpi_n : \prod_{n\in\mathbb{N}} F_n \to \mathbb{R}$ defined by

$$\left(\bigoplus_{n\in\mathbb{N}} \varpi_n\right)\left(\sum_{j=1}^{m}\prod_{k=1}^{n_j}(f_{k,j}\circ\pi_k)\right) := \sum_{j=1}^{m}\prod_{k=1}^{n_j}\varpi_k(f_{k,j}),$$

and then uniquely extended to the product topology according to the extension theorem, is in $\mathrm{Char}^*(\prod_{n\in\mathbb{N}} F_n)$. Every element of $\mathrm{Char}^*(\prod_{n\in\mathbb{N}} F_n)$ is generated by a function defined on the base $\bigoplus_{n\in\mathbb{N}} F_n$ as above. Since the space $\mathcal{F}_n$ is c-compact, for every $n \in \mathbb{N}$, we get that $\prod_{n\in\mathbb{N}} F_n$ is c-compact. □

Proposition

Let $\mathcal{F} = (X, F)$ be a c-compact Bishop space, $\mathcal{G} = (Y, G)$ a Bishop space and $T : F \to G$ a non-zero ring homomorphism. There exists a mapping $\tau : Y \to X \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ such that $T(f) = f \circ \tau$, for every $f \in F$.

Theorem (Banach-Stone theorem for c-compact spaces)

Let $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ be c-compact Bishop spaces. If the rings F, G are isomorphic, then the Bishop spaces $\mathcal{F}, \mathcal{G}$ are isomorphic.

Theorem

Suppose that $\mathcal{F} = (X, F)$ is a c-compact space and $D \subseteq X$ is F-dense and embedded in X. If $\mathcal{G} = (Y, G)$ is c-compact and $h : D \to Y \in \mathrm{Mor}(\mathcal{F}_{|D}, \mathcal{G})$, there exists a unique mapping $\tilde{h} : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ such that $\tilde{h}_{|D} = h$.

Bishop proved in a highly technical way the c-compactness of a compact metric space, based on the theory of normed spaces.

Proposition (Bishop)

Let K be a compact metric space, and Γ the set of all non-zero bounded multiplicative linear functionals on $C_u(K, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$, or $\mathbb{C}$. Then every element of Γ is of the form π_x for some $x \in K$.

Proposition

If (X, d) is a compact metric space, then $\mathrm{ker}(\pi)$ is a located subset of $C_u(X)$, for every $\pi \in \mathrm{Char}^*(C_u(X))$. Moreover, there is at most one $x \in X$ such that $d_x \in \mathrm{ker}(\pi)$, and if there is $x \in X$ such that $d_x \in \mathrm{ker}(\pi)$, then $\pi = \pi_x$.

Theorem

Let $a, b \in \mathbb{R}$ such that $a < b$, and let $\mathrm{id}_{[a,b]}$ be the identity on $[a, b]$.
(i) $C_u([a, b]) = \bigvee \mathrm{id}_{[a,b]}$.
(ii) $C_u([a, b])$ is c-compact.

Proof.

(i) Let $\phi : [a, b] \to \mathbb{R} \in C_u([a, b])$. Since $\phi([a, b]) \subseteq I$, where I is some compact interval of $\mathbb{R}$, by Bishop's version of the Tietze extension theorem there is some $\tilde{\phi} \in \mathrm{B}(\mathbb{R})$ which extends ϕ. Hence $\phi = \tilde{\phi} \circ \mathrm{id}_{[a,b]} \in \bigvee \mathrm{id}_{[a,b]}$.
(ii) If $\pi \in \mathrm{Char}^*(C_u([a, b]))$, let $\pi(\mathrm{id}_{[a,b]}) = w$. Since $\overline{a} \leq \mathrm{id}_{[a,b]} \leq \overline{b}$, we have that $a = \pi(\overline{a}) \leq w \leq \pi(\overline{b}) = b$, therefore

$$\pi(\mathrm{id}_{[a,b]}) = w = \mathrm{id}_{[a,b]}(w).$$

By (i) we get that $\pi = \pi_w$. □

Definition

The *Hilbert cube* $\mathcal{I}^{\infty}$ is the Bishop space

$$\mathcal{I}^{\infty} := (I^{\infty}, (\mathrm{B}(\mathbb{R}))^{\mathbb{N}}{}_{|I^{\infty}}),$$

$$I^{\infty} := \{(x_n)_{n=1}^{\infty} \in l^2(\mathbb{N}) \mid \forall_{n\in\mathbb{N}}(|x_n| \leq \frac{1}{n})\},$$

$$l^2(\mathbb{N}) := \{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} x_n^2 < \infty\}.$$

Corollary

(i) The Hilbert cube is isomorphic to $\mathcal{I}_{(-1)1}^{\mathbb{N}}$*, where* $\mathcal{I}_{(-1)1} = ([-1,1], C_u([-1,1]))$*.*

(ii) The Hilbert cube is a c-compact Bishop space.

Proposition

If (X, d) is a compact metric space, then $\mathcal{U}(X) = (X, C_u(X))$ is topologically embedded into the Hilbert cube.

Remark

The topological embedding e of a compact metric space X into the Hilbert cube is in $\mathrm{Lip}(X, [0,1]^{\mathbb{N}}, 1)$.

Consequently, e is uniformly continuous. This follows also directly from Bridges's backward uniform continuity theorem.

Next we show that a compact space with the uniform topology is a c-compact space i.e., c-**compactness generalizes metric compactness**.

Comfort didn't show this result.

Definition

If $(X, d), (Y, \rho)$ are metric spaces, a mapping $f : X \to Y$ is called *hyperinjective*, or a *hyperinjection*, if for any two compact subsets A, B of X with $\inf\{d(a, b) \mid a \in A, b \in B\} > 0$, there exists $r > 0$ such that $\rho(f(a), f(b)) \geq r$, for every $a \in A$ and $b \in B$.

Proposition (Bishop-Bridges)

Let $(X, d), (Y, \rho)$ be metric spaces and let $f : X \to Y$ be uniformly continuous and hyperinjective. If X is compact, then the inverse map $g : f(X) \to X$ is uniformly continuous and hyperinjective on $f(X)$, and $f(X)$ is compact.

Lemma

The topological embedding e of a compact metric space X into the Hilbert cube is hyperinjective.

Theorem

If (X, d) is a compact metric space, the Bishop space $(X, C_u(X))$ is c-compact.

Proposition

Let $\mathcal{F} = (X, F)$ be a a Bishop space such that F separates the points of X.
(i) $\mathcal{F}$ is pseudo-compact space if and only if it is topologically embedded in the product space of closed intervals $\prod_{f \in F} \mathcal{M}_f$, where $\mathcal{M}_f = ([-M_f, M_f], C_u([-M_f, M_f])$ and $M_f > 0$ is a fixed bound for $f \in F$.

(ii) If $F = \bigvee F_0$, then $\mathcal{F}$ is pseudo-compact space if and only if it is topologically embedded in the product space of closed intervals $\prod_{f_0 \in F_0} \mathcal{M}_{f_0}$, where $\mathcal{M}_{f_0} = ([-M_{f_0}, M_{f_0}], C_u([-M_{f_0}, M_{f_0}]))$ and $M_{f_0} > 0$ is a fixed bound for $f_0 \in F_0$.

Definition

Let $\mathcal{F} = (X, F)$ be a pseudo-compact Bishop space such that F separates the points of X and let e be the topological embedding of $\mathcal{F}$ in $\prod_{f \in F} \mathcal{M}_f$.

The **Stone-Čech** c**-compactification of** $\mathcal{F}$ is the Bishop space $\beta\mathcal{F} = (\beta X, \beta F)$, where $\beta X = c(e(X))$, the c-closure of $e(X)$, and

$$\beta F = (\bigvee_{f \in F} \mathrm{pr}_f)_{|c(e(X))}.$$

If $\mathcal{F}_0 = (X, \bigvee F_0)$ is a pseudo-compact Bishop space such that $\bigvee F_0$ separates the points of X and e_0 is the topological embedding of $\mathcal{F}_0$ in $\prod_{f_0 \in F_0} \mathcal{M}_{f_0}$, the **Stone-Čech** c**-compactification of** $\mathcal{F}_0$ is the Bishop space $\beta\mathcal{F}_0 = (\beta X, \beta \bigvee F_0)$, where $\beta X = c(e_0(X))$ and

$$\beta \bigvee F_0 = (\bigvee_{f_0 \in F_0} \mathrm{pr}_{f_0})_{|c(e_0(X))}.$$

Proposition

(i) The Stone-Čech c-compactification $\beta\mathcal{F} = (\beta X, \beta F)$ *of of a pseudo-compact Bishop space* $\mathcal{F} = (X, F)$ *with separating topology* F *is c-compact.*
(ii) The Stone-Čech c-compactification $\beta\mathcal{F}_0 = (\beta X, \beta \bigvee F_0)$ *of of a pseudo-compact Bishop space* $\mathcal{F}_0 = (X, \bigvee F_0)$ *with separating topology* $\bigvee F_0$ *is c-compact.*

Proposition

Suppose that $\mathcal{F} = (X, F)$ *is a pseudo-compact space and* F *is separating.*
(i) If $\mathcal{G} = (Y, G)$ *is a c-compact space and* $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$*, there exists a unique* $h^{\beta} : \beta X \to Y \in \mathrm{Mor}(\beta\mathcal{F}, \mathcal{G})$ *such that* $(h^{\beta})_{|X} = h$*, where the expression* $(h^{\beta})_{|X} = h$ *is short for* $h^{\beta} \circ e = h$ *and* e *is the topological embedding of* $\mathcal{F}$ *in* $\beta\mathcal{F}$*.*
(ii) For every $f \in F$ *there exists a unique* $f^{\beta} \in \beta F$ *such that* $(f^{\beta})_{|X} = f$*, where the expression* $(f^{\beta})_{|X} = f$ *is short for* $f^{\beta} \circ e = f$*.*

Definition

Let $\mathcal{F} = (X, F)$ be a pseudo-compact Bishop space such that F separates the points of X. A c-**compactification of** $\mathcal{F}$ is a pair $(\mathcal{G}, \varepsilon)$, where $\mathcal{G} = (Y, G)$ is a c-compact space and $\varepsilon : X \to Y$ is a topological embedding of $\mathcal{F}$ into $\mathcal{G}$ such that $\varepsilon(X)$ is G-dense in Y.

Proposition (Uniqueness of Stone-Čech c-compactification)

Let $\mathcal{F} = (X, F)$ be a pseudo-compact Bishop space such that F separates the points of X. If $(\mathcal{G}, \varepsilon)$ is a c-compactification of $\mathcal{F}$ such that for every c-compact Bishop space $\mathcal{H} = (H, Z)$ and $\theta : X \to Z \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})$, there exists a unique mapping $\theta^G : Y \to Z \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$ such that $(\theta^G)_{|X} = \theta$ i.e., $\theta^G \circ \varepsilon = \theta$, then there exists $i : \beta X \to Y$ an isomorphism between $\beta\mathcal{F}$ and $\mathcal{G}$ such that $i \circ e = \varepsilon$.

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