Interval numbers in point-free topology: localic suplattices and positivity relations

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Objectives:

- Describe some possible topologies on interval numbers...
- ...and their point-free counterparts.

Motivations:

- "Always topologize!"
- "I want to break *point*-free..."
  (see Tychonoff thm,..., Alex Simpson's measure)
- ...after all, point-free topology it's the topic I know best!

What is point-free topology?

Example: the reals

$\Omega\mathbb{R} = \{A \subseteq \mathbb{R} \mid A \textit{ is open}\}$ is a **frame** w.r.t. $\subseteq$

(frame = complete lattice s.t. $\wedge$ distributes over $\bigvee$)

(CLASS) $\mathbb{R} \cong \{\text{completely prime filters of } \Omega\mathbb{R}\}$

More generally:

locale = a frame that claims to consist of the opens of a space

localic map = a thing that claims to be the preimage of a continuous function

point = a completely prime filter

Point-free VS point-wise topology

There is an adjunction between **topological spaces** and **locales**.
There is an equivalence between **sober** topological spaces and **spatial** locales.

sober = space of points of a locale =
= every cp-filter is the neighbourhood filter of a unique point

(Hausdorff $\Rightarrow$ sober)

spatial = frame of open of a topological space

- $\{$*sober spaces*$\} \hookrightarrow \{$*locales*$\}$
- If X is a T2 space, then $\{$*subspaces of* $X\} \subseteq \{$*sublocales of* $X\}$.
  Example: $\mathbb{R}$ has (many) more sub-*locales* than sub-*spaces*.

Predicative point-free topology: **formal topology**

$\Omega\mathbb{R}$ has a base of open intervals with rational endpoints.

All the information about $\Omega\mathbb{R}$ can be coded by:

- the set $S = \{(a,b) \in \mathbb{Q} \times \mathbb{Q} \mid a < b\}$ and
- a *cover* relation $(a,b) \lhd \{(x_i, y_i) \mid i \in I\}$ which say when $]a,b[\ \subseteq \bigcup_{i \in I}]x_i, y_i[$

More generally:

formal topology $=$ a locale with a base $=$ a cover relation $(S, \lhd)$

formal map $=$ the relation induced between the two bases by a localic map

formal point $=$ the intersection of the base and a completely prime filter

The formal topology of interval numbers

There exist a cover relation $\lhd_{\mathbb{IR}}$ on the set $S = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} \mid a < b\}$ s.t.

$$(\textit{classically}) \qquad \mathbb{IR} \cong \{\textit{formal points of } (S, \lhd_{\mathbb{IR}})\}$$

(S. Negri, 2002)

- The topology on $\mathbb{IR}$ induced by $(S, \lhd_{\mathbb{IR}})$ is the Scott-topology.
- The *specialization* order is just $\supseteq$.
- $(\mathbb{IR}, \supseteq)$ is a **continuous domain** $=$
- $=$ dcpo (all directed sups) $+$ every x is a directed join of y's *way-below* x.
- $([y_1, y_2] << [x_1, x_2]$ iff $[y_1, y_2] \supset [x_1, x_2])$

More generally. . .

Point-free topology and continuous domains

- The "points" of a point-free topology form a **dcpo** (wrt specialization).

  (Johnstone)

- The topology on the points is coarser than the Scott-topology.

  (Abramsky&Jung)

- Every **continuous domain** can be represented as the space of points of a formal topology...

  which is a constructive version of Abramsky&Yung's:

  a continuous domain equipped with the Scott-topology is a sober spaces.

Point-free interval analysis

Let $\mathcal{R}$ and $\mathrm{I}\mathcal{R}$ be the point-free versions of $\mathbb{R}$ and $\mathrm{I}\mathbb{R}$.

Then:

- $\mathcal{R}$ embeds in $\mathrm{I}\mathcal{R}$ and
- any morphism from $\mathcal{R}$ to $\mathcal{R}$ lifts to a morphism from $\mathrm{I}\mathcal{R}$ to $\mathrm{I}\mathcal{R}$
- (e. g. the arithmetic operations on $\mathcal{R}$ lift to the ordinary interval arithmetic operations).

(A. Hedin's PhD thesis 2011)

Toward a different perspective...

Let $\mathrm{C}\mathbb{R}$ be $\{C \subseteq \mathbb{R} \mid C \textit{ closed}\}$

$\mathrm{I}\mathbb{R}$ is a subspace of $\mathrm{C}\mathbb{R}$... if we put a topology on the latter.

Lower (Vietoris) hypertopology
subbase: $\{\diamond A \mid A \subseteq \mathbb{R} \textit{ open}\}$ where $\diamond A = \{C \in \mathrm{C}\mathbb{R} \mid C \between A\}$

Classically:
that is the upper interval topology (aka weak topology) on the poset $(\mathrm{C}\mathbb{R}, \subseteq) =$
$=$ the coarsest topology s.t. $\subseteq$ is the specialization order $=$
$=$ the coarsest topology s.t. every $\{X \in \mathrm{C}\mathbb{R} \mid X \subseteq C\}$ is closed, for C closed in $\mathbb{R}$.

About $\mathrm{I}\mathbb{R}$ with the subspace topology

- The specialization order is $\subseteq$ and
- hence the topology is finer than the weak topology wrt $\subseteq$
- in fact, it is strictly finer than that:
  $\{[x, y] \in \mathrm{I}\mathbb{R} \mid x < 0\}$ is open in the induced topology, not in the weak one.

- Actually, it is just the Scott topology.

- Is it sober?

The lower powerlocale $\mathrm{P_L}\mathcal{R}$

The point-free version of the lower hyperspace over the reals is $\mathrm{P_L}\mathcal{R}$, the *lower powerlocale* of $\mathcal{R}$.

(cf. the Hoare powerdomain)

- Its underlying frame is generated by the $\diamond A$'s.
- Actually it is the free frame over $\Omega\mathbb{R}$ qua suplattice (because $\diamond$ preserve unions, and that's it).

- As a formal topology, it is of the form $(Fin(S), \lhd)$ where $S = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} \mid a < b\}$ as before and $Fin(S)$ is the set of (Kuratowski-)finite subsets of S.

- Its points are the closed subset of $\mathbb{R}$, classically. Constructively, they correspond to *overt*, *weakly closed* sublocales of $\mathcal{R}$.

Positivity relations

Giovanni Sambin (late 90's) introduced formal topologies with *positivity relations*.

$$(S, \lhd, \ltimes)$$

These objects are called *positive topologies* (or *balanced formal topologies*).

- $\ltimes$ has the same logical type as $\lhd$
- $\ltimes$ and $\lhd$ have dual properties (almost always)
- $\ltimes$ corresponds to a family of distinguished "closed sets"
  (actually a sub-suplattice of all possible overt, weakly closed sublocales)
- Many $\ltimes$'s exist which are compatible with a given $\lhd$.

Localic suplattices

From a localic point of view,

- each positive relation on a locale X corresponds to a *localic suplattice*, that is, an algebra for the lower powerlocale monad $\mathrm{P_L}$;
- moreover, it is a sub-object of $\mathrm{P_L}X$ in the Eilenberg-Moore category for $\mathrm{P_L}$.

Positivity relations on X = (spatial) localic sub-suplattices of $\mathrm{P_L}X$

[F.C. - Steve Vickers "*Positivity relations on a locale*" APAL 2016]

A different point-free perspective on interval numbers

At last!

Idea: use a suitable positivity relation to single out the interval numbers.

1. Start from $\mathrm{I}\mathbb{R}$.
2. Break free of points: $\mathrm{I}^+\mathbb{R} = \{[x, y] \mid x < y\}$
3. Make it into a dcpo: $(\mathrm{I}^+\mathbb{R}) = \{$*closed intervals of positive or infinite length*$\}$

   (cf. Kulisch's complete interval arithmetic)
4. Add finite joins: $reg\,\mathrm{C}\mathbb{R} = \{C \in \mathrm{C}\mathbb{R} \mid C = cl(int(C))\} = \{cl(A) \mid A \in \Omega\mathbb{R}\}$
5. which is a sub-suplattice of $\mathrm{C}\mathbb{R}$

   (the least sub-suplattice of $\mathrm{C}\mathbb{R}$ which contains $\mathrm{I}^+\mathbb{R}$)

By F.C.&S.Vickers 2016, $reg\,\mathrm{C}\mathbb{R}$ corresponds to a positivity relation on $\mathcal{R}$.

Explicitly...

For $S = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} \mid a < b\}$, $(a, b) \in S$ and $U \subseteq S$

$$(a, b) \ltimes U \quad iff \qquad \exists (c, d) \in S. (a, b) \in \diamond(c, d) \subseteq U$$

where $\diamond(c, d) = \{(x, y) \in S \mid (x, y) \between (c, d)\} = \{(x, y) \in S \mid x < d \,\&\, c < y\}$.

If $\lhd$ is the usual cover for the reals, then $(S, \lhd, \ltimes)$ is a structure in which

- $\lhd$ gives us access to the reals and
- $\ltimes$ gives us access to a family of distinguished sublocales (which are the regular closed subsets of $\mathbb{R}$, classically).

So reals and positive-length intervals live in two separate parts of the same structure; this makes sense constructively, since you are not able to decide whether $[x, y]$ is 0-length or positive-length!

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