Effectively bi-immune sets and randomness

Mushfeq Khan
with Achilles Beros and Bjørn Kjos-Hanssen

University of Hawai'i at Mānoa

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A review of some terms

Definition

An infinite set $X \subseteq \omega$ is said to be immune if it does not contain any infinite r.e. subset.

Definition

An infinite set $X \subseteq \omega$ is said to be effectively immune if there is a recursive function f such that for all e, if W_e is a subset of X, then $|W_e| \leq f(e)$.

The study of sets whose immunity is witnessed in an effective manner dates back to Post.

Effective immunity in the co-r.e. sets

Post (1944) called an r.e. set whose complement is immune *simple*. He provided a construction in the hope that it could be shown or adapted to produce a Turing-incomplete nonrecursive r.e. set.

Smullyan (1964) observed that Post's simple set is actually *effectively simple*, i.e., its complement is effectively immune.

In fact, one has to try hard to produce a simple set that is not effectively simple, and such sets were constructed implicitly by Friedberg (1957) and Muchnik (1956), and explicitly by Sacks (1964), via priority arguments.

Soon after, Martin (1966) showed, in a result later generalized by the Arslanov Completeness Criterion, that every effectively simple set is complete.

Effective immunity outside the co-r.e. sets

The notion of effective immunity turned out to be an important one outside the context of the co-r.e. sets:

Theorem (Jockusch, 1989)

Every DNR *degree contains an effectively immune set.*

This, together with an earlier observation by Arslanov, Nadirov, and Solov'ev (1977), showed that the Turing degrees of the DNR functions and those of the effectively immune sets coincide.

A set is *bi-immune* if both it and its complement are immune.

Theorem (Jockusch and Lewis, 2013)

Every DNR *function computes a bi-immune set.*

The same authors asked:

Question

Call a set *effectively bi-immune* (or EBI) if both it and its complement are effectively immune. Does every DNR function compute an effectively bi-immune set?

When the function f witness the effective immunity of both X and its complement, we say X *is effectively immune via* f.

Theorem (Beros, 2015)

There is a DNR *function that computes no* EBI *set.*

To summarize, every DNR function computes an effectively immune set (in fact, of the same degree), and a bi-immune set. But not necessarily an effectively bi-immune set.

Effective bi-immunity is a measure-typical property: every Martin-Löf random real is effectively bi-immune. However, weaker levels of randomness like Schnorr and computable randomness do not suffice.

DNR functions and effectively bi-immune sets

Beros's proof is a direct construction involving some intricate combinatorics, but it also follows from an existing result and the following observation:

Proposition

Every EBI *set computes a recursively bounded* DNR *function.*

Proof

- Suppose X is effectively bi-immune via f.
- Let h be a recursive function such that for all n, $W_{h(n)}$ is the finite set coded by $\varphi_n(n)$, if it converges.
- Let $g(n)$ be the code for the set consisting of the first $f(h(n)) + 1$ elements of X.
- Let $\bar{g}(n)$ be the code for the set consisting of the first $f(h(n)) + 1$ elements of $\bar{X}$.
- g and $\bar{g}$ are DNR.
- Now let $g^* = \min(g, \bar{g})$. Clearly, it is DNR. Moreover, the largest element in the finite set coded by $g^*(n)$ is bounded by $2(f(h(n))) + 1$.

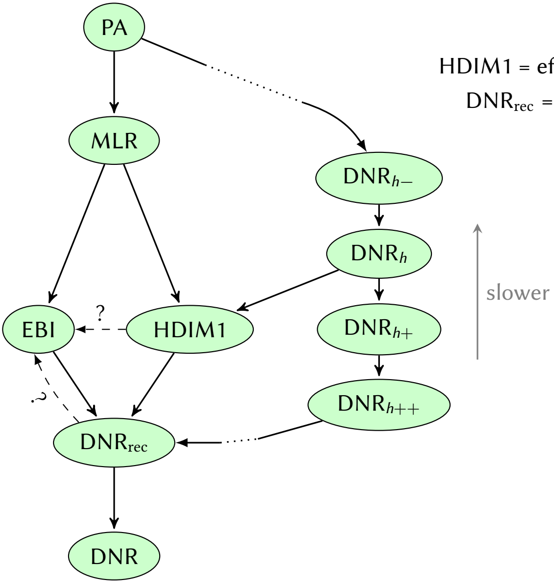
It's enough now to appeal to the following result:

Theorem (Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman, 2004)

There is a DNR *function that computes no recursively bounded* DNR *function.*

This DNR function cannot compute an EBI set.

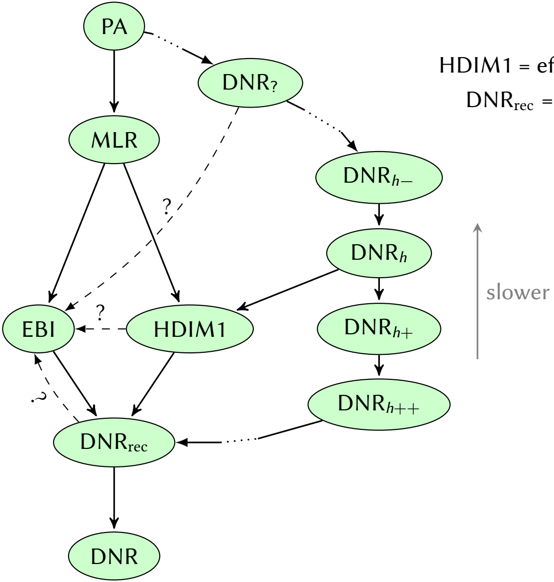
Implications



$\rightarrow$ means $\geq_w$

HDIM1 = effective Hausdorff dimension 1

$\mathrm{DNR}_{\mathrm{rec}}$ = recursively bounded DNRs



$\to$ means $\geq_w$

HDIM1 = effective Hausdorff dimension 1

DNR$_{rec}$ = recursively bounded DNRs

By an *order function* we mean a recursive, nondecreasing, and unbounded function that takes on values greater or equal to 2.

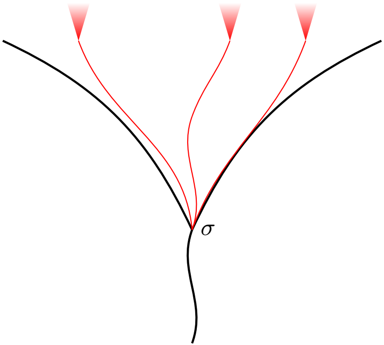
Theorem

Let q be any order function. Then for any oracle X, there is a q-bounded DNR *function relative to X that computes no* EBI *set.*

Construct a q-bounded DNR g by forcing with conditions of the form (σ, B) where

- $\sigma \in q^{<\omega}$ and $B \subset q^{<\omega}$, and
- B is $q(|\sigma|)$-small above σ.

B is a set of "bad" strings that we cannot extend. There is no effectivity assumption on it.



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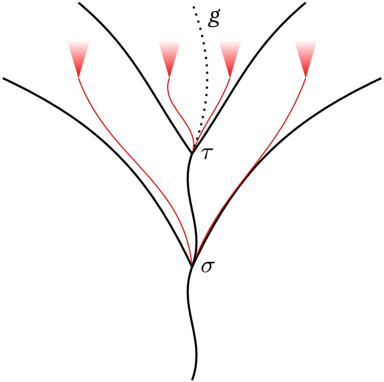
Given a reduction Γ and a recursive function h, we want to extend σ to a string $\tau \notin B$ so as to force the fact that Γ^g is not an h-EBI.



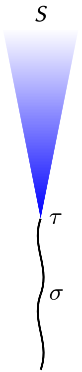
EBI and slow-growing DNRs (sketch)

We cannot force Γ to be total, but we can make sure that it converges "a lot".

We argue that there exists an extension τ of σ and an infinite tree S, bushy "enough" above τ such that Γ is *i.o.-constant* on S:

- For infinitely many n, $\Gamma^f(n)$ is constant as f ranges over $[S]$.

S isn't effective.



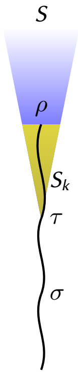
EBI and slow-growing DNRs (sketch)

But because it exists, uniformly in k, we can *effectively* find a *finite* tree S_k, such that for k distinct inputs, Γ^ρ is constant as ρ ranges over the leaves of S_k.

For $i \in \{0, 1\}$, the set of inputs such that the constant output is i is an r.e. set.

Use the recursion theorem to choose k large enough so that one of these is bigger than the bound h allows.

Extend τ to a "good" leaf ρ of S_k to force this r.e. set into Γ^g (or $\overline{\Gamma^g}$).



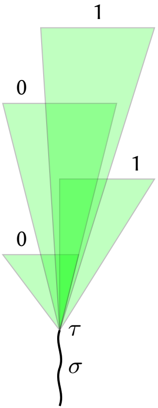
EBI and slow-growing DNRs (sketch)

The noneffective argument for the existence of S splits into two cases.

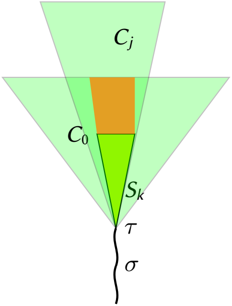
The more involved of these is one where there is a good extension τ of σ above which Γ infinitely often favors one output heavily over the other.

More precisely, for infinitely many inputs n, the set of ρ extending τ such that $\Gamma^{\rho}(n) = i$ is much bigger than the set of those such that $\Gamma^{\rho}(n) = 1 - i$.

These "dominant majority" sets don't necessarily form a *coherent* sequence when viewed as trees. But their pairwise intersections are very large.



This allows us to inductively construct S. Given S_k and an infinite collection of majority trees $C_0, C_1, C_2, \ldots$, all of which contain S_k, we consider how $C_0 \cap C_j$ extends S_k.



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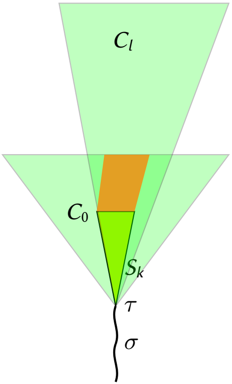

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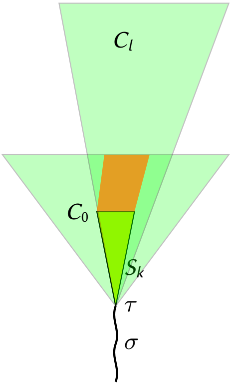
$C_0 \cap C_l$ possibly extends S_k differently.

But since C_0 is finite, for infinitely many $j > 0$, $C_0 \cap C_j$ gives the *same* extension of S_k. Keep these majority trees, which "elect" the same extension of S_k to S_{k+1}, and discard the rest.



Corollary

There is a real of effective Hausdorff dimension 1 that computes no EBI *set.*

Compare this to:

Theorem (Greenberg and Miller, 2011)

There is a real of effective Hausdorff dimension 1 that computes no MLR set.

An open question

From this perspective, **EBI** appears to be closer than expected to **MLR**.

Question

Does every EBI compute an MLR set?

We know that there is an EBI Turing degree that contains no MLR set: First, the join of two EBI sets is EBI. Second, every PA degree contains the join of two MLR sets (Barmpalias, Lewis, and Ng, 2010), and therefore, an EBI set. In particular, any incomplete PA degree contains an EBI set, and cannot contain an MLR set, by a theorem of Stephan.

Thank you