Computable analysis and reverse mathematics

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joint work with Andre Nies and Marcus Triplett

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Introduction

- Results in computable analysis often can be re-understood in reverse mathematics.
- Actually, relativized versions of the statements almost indicate the corresponding reverse math results.

$$\begin{array}{rcl} X' & \Leftrightarrow & \mathrm{ACA}_0 \\ \text{PA-degree rel. to } X & \Leftrightarrow & \mathrm{WKL}_0 \\ \text{ML-random rel. to } X & \Leftrightarrow & \mathrm{WWKL}_0 \end{array}$$

Here, we will see some case studies of this in the study of bounded variation functions.

- Jordan's decomposition theorem for bounded variation functions.
- Lebesgue's theorem on the differentiability of bounded variation functions.

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- Jordan's decomposition theorem for bounded variation functions.
- Lebesgue's theorem on the differentiability of bounded variation functions.

Bounded variation functions on $[0, 1]$

We mainly deal with bounded variation functions on $[0, 1] \cap \mathbb{Q}$ coded by the following way.

Definition

A (code for a) rationally presented function is a pair $f = (Z_f, r_f)$ where $Z_f : [0, 1] \cap \mathbb{Q} \times \mathbb{Q} \to 2$ and $r_f \in \mathbb{R}$ such that

- $Z_f(x, p) \leq Z_f(x, q)$ for any $p \leq q$,
- for any $x \in [0, 1] \cap \mathbb{Q}$ there exist $p, q \in \mathbb{Q}$ such that $Z_f(x, p) = 0$ and $Z_f(x, q) = 1$.

$f : [0, 1]_{\mathbb{Q}} \to \mathbb{R}$ is defined as $f(x) = r_f + \sup\{p : Z_f(x, p) = 0\}$.

Note that

- if f is computable, then $f(x)$ is computable.
- above definition can be made within RCA_0.

Bounded variation functions on $[0,1]$

Definition

A rationally presented function f is said to be of bounded variation if if there is $k \in \mathbb{N}$ such that $S(f,\Pi) \leq k$ for every partition Π of $[0,1]$, where

$$\Pi = \{0 = t_0 \leq \cdots \leq t_n = 1\} \subseteq [0,1] \cap \mathbb{Q},$$

$$S(f,\Pi) = \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|.$$

We can deal with continuous functions of bounded variation within RCA_0 based on the following.

Proposition (RCA_0)

Every continuous function $f : [0,1] \to \mathbb{R}$ has a rational presentation on $[0,1] \cap \mathbb{Q}$.

Contents

First example

Theorem

The following are equivalent over RCA_0*.*

1. WKL_0.
2. *For every rationally presented function f of bounded variation, there is a rationally presented non-decreasing function $g : [0,1]_{\mathbb{Q}} \to \mathbb{R}$ such that $f \leq_{\mathrm{slope}} g$.*

Here, we let

$$f \leq_{\mathrm{slope}} g \text{ iff } \forall x, y \in [0,1]_{\mathbb{Q}}[x < y \to (f(y) - f(x) \leq g(y) - g(x))].$$

Note that the second clause is the Jordan decomposition theorem: $f = g - (g - f)$ where both of g and $g - f$ are non-decreasing.

First example

Proof of 1 → 2: easy.
It is a straightforward formalization of the following theorem within WKL_0.

Theorem (essentially Brattka/Miller/Nies 2011)

Let **a** *be a PA-degree. Then, for any computable rationally presented function f of bounded variation, there exists a rationally presented function* $g \leq_T \mathbf{a}$ *such that* $f \leq_{\text{slope}} g$.

- Let k be the bound of the variation of f.
- $P := \{g : f \leq_{\text{slope}} g, 0 \leq g \leq k\}$ is a non-empty Π^0_1-class.
- PA-degree can compute a member of P.

  $\Rightarrow$ one can find a member of P by WKL.

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Proof of 2 → 1.

We will formalize the following theorem within RCA_0.

Theorem (Greenberg/Miller/Nies 2013, in preparation)

There exists a computable function f of bounded variation on $[0, 1]$ such that any rationally presented function $g \geq_{\text{slope}} f$ computes a PA-degree.

For a given tree $T \subseteq 2^{<\mathbb{N}}$, put $[T] = \{\sum_{n \in X} 2^{-n-1} : X \text{ is a path of } T\}$.

- For a given infinite computable tree T with no computable path, one can construct a computable function f of bounded variation on $[0, 1]$ such that

  "if $g \geq_{\text{slope}} f$ and g is continuous on $[T]$, then g computes $\mathbf{0}'$".
- If g is/is not continuous on $[T]$...

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  (In fact, any c.e. set can be coded.)
- If g is continuous on $[T]$, then g computes $\mathbf{0}'$, thus it computes a path of T.
- If g is not continuous on $[T]$, then there exists $q > 0$ such that $P := \{z \in [T] : \forall x, y \in [0, 1] \cap \mathbb{Q}(x < z < y \to g(y) - g(z) \geq q)\}$ is not empty.
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Within RCA_0, we can work in a parallel way.
(We will directly code $\mathrm{NExt}(T) := \{\sigma \in T : \sigma$ is non-extendible$\}$ to computes a path of T.)

- For a given infinite tree T with no path, one can construct a continuous function f of bounded variation on $[0,1]$ such that "if $g \geq_{\text{slope}} f$ and g is continuous on $[T]$, then g computes $\mathrm{NExt}(T)$".
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Caution!

We need the following well-known theorem.

Theorem

Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite computable tree. If T has at most finitely many paths, then T has a computable path.

Question

How can we understand this situation in reverse mathematics?

- "Any infinite tree $T \subseteq 2^{<\mathbb{N}}$ which has at most finitely-many paths has a path" is already equivalent to WKL since $\neg\mathrm{WKL}$ implies the existence of an infinite tree with no path.
- Thus, we will consider several structural conditions to support the finiteness of paths.

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We will consider the following versions of WKL.

1. $\mathrm{WKL}(\textit{ext-bd})$: an infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|T^{=n}_{\mathrm{ext}}| \leq c$, where $T^{=n}_{\mathrm{ext}} = \{\sigma \in T \mid \mathrm{lh}(\sigma) = n \wedge \sigma \text{ is extendible}\}$.
2. $\mathrm{WKL}(\textit{w-bd})$: an infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|T^{=n}| \leq c$, where $T^{=n} = \{\sigma \in T \mid \mathrm{lh}(\sigma) = n\}$.
3. $\mathrm{WKL}(\textit{pf-bd})$: an infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any prefix-free set $P \subseteq T$, $|P| \leq c$.

* For a fixed standard $c \in \omega$, they are all provable within RCA_0.

Note that WKL restricted to a tree with at most finitely many paths are studied in various context, e.g., in Weihrauch degrees, constructive math,...

Non-trivial induction strength

$\mathrm{WKL}(\mathit{pf}\text{-}\mathit{bd})$, $\mathrm{WKL}(\mathit{w}\text{-}\mathit{bd})$, $\mathrm{WKL}(\mathit{ext}\text{-}\mathit{bd})$ are all true in ω-model of RCA_0.

Theorem

1. $\mathrm{WKL}(\mathit{w}\text{-}\mathit{bd})$ *and* $\mathrm{WKL}(\mathit{ext}\text{-}\mathit{bd})$ *are equivalent.*
2. $\mathrm{WKL}(\mathit{w}\text{-}\mathit{bd})$ *is provable in* $\mathrm{RCA}_0 + \mathrm{WKL} \vee \mathrm{I}\Sigma^0_2$.
3. WWKL_0 *does not imply* $\mathrm{WKL}(\mathit{w}\text{-}\mathit{bd})$.
4. $\mathrm{WKL}(\mathit{w}\text{-}\mathit{bd})$ *plus* $\exists X \forall Y (Y \leq_T X)$ *implies* $\mathrm{I}\Sigma^0_2$.

So, $\mathrm{WKL}(\mathit{w}\text{-}\mathit{bd})$ is still too strong to use within RCA_0 because of the lack of induction.

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$\mathrm{WKL}(\mathit{pf}\text{-}\mathit{bd})$ *is provable in* RCA_0.

Thus, we need to use this version.

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Within RCA_0, we can work in a parallel way.

- For a given infinite tree T with no path, one can construct a continuous function f of bounded variation on $[0,1]$ such that "if $g \geq_{\text{slope}} f$ and g is continuous on $[T]$, then g computes $\mathrm{NExt}(T)$".
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- We will approximate P by a tree with the size of prefix-free subsets bounded.

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Second example

The upper and lower pseudo-derivatives of f are defined by

$$\bar{D}f(x) = \lim_{h \to 0^+} \sup \left\{ \frac{f(b)-f(a)}{b-a} : a \leq x \leq b \ \wedge \ 0 < b - a < h \right\}, \text{ and}$$

$$\underline{D}f(x) = \lim_{h \to 0^+} \inf \left\{ \frac{f(b)-f(a)}{b-a} : a \leq x \leq b \ \wedge \ 0 < b - a < h \right\}.$$

A function f is pseudo-differentiable at $z \in (0, 1)$ if $\underline{D}f(z)$ and $\bar{D}f(z)$ are both finite and equal.

Theorem

The following are equivalent over RCA_0.

1. WWKL_0
2. *Every rationally presented function of bounded variation is pseudo-differentiable at some point.*
3. *Every rationally presented function of bounded variation is pseudo-differentiable almost surely.*

Second example

Proof of 2 → 1: easy.
It is a straightforward formalization of the following within RCA_0.

Theorem (Brattka/Miller/Nies 2011)

There is a computable function f of bounded variation on $[0, 1]$ such that $f'(z)$ exists only for Martin-Löf random reals z.

- Given a ML-test $\{U_i\}_{i\in\mathbb{N}}$, one can construct a computable function of bounded variation f such that f is not (pseudo-)differentiable at any $z \in \bigcap U_i$.

Within RCA_0, given a tree T such that $[T]$ has a positive measure,

- One can construct a ML-test $\{U_i\}_{i\in\mathbb{N}}$ (rel. to T) so that any $z \in \bigcap U_i$ computes a path of T.
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Second example

Proof of 1 $\rightarrow$ 2. (One can show 1 $\rightarrow$ 3 in a similar way.)
We will formalize the following theorem within WWKL_0.

Theorem (essentially Brattka/Miller/Nies 2011)

Every computable rationally presented function f of bounded variation is differentiable at any Martin-Löf random real z.

- Every non-decreasing computable rationally presented function f_0 is differentiable at any Martin-Löf random real z (actually, computably random is enough).
- z is ML-random iff it is ML-random relative to a PA-degree $\mathbf{a}$.
- By Jordan decomposition, there exist non-decreasing functions $g, h \leq_T \mathbf{a}$ such that $f = g - h$.
- f is differentiable at z since g and h are differentiable at z.

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Within WWKL_0, the proof won't work directly...

- Every non-decreasing computable rationally presented function f_0 is differentiable at any Martin-Löf random real z.

  $\Rightarrow$ this is formalizable within RCA_0.
- z is ML-random iff it is ML-random relative to a PA-degree $\mathbf{a}$.

  $\Rightarrow$ want a PA-degree with preserving randomness!
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We will first formalize/modify the first and second steps.

- Every non-decreasing computable rationally presented function f_0 is differentiable at any Martin-Löf random real z.

Lemma

RCA_0 *proves the following.*

every non-decreasing rationally presented function f_0 *is pseudo-differentiable at any Martin-Löf random real* z.

- z is ML-random iff it is ML-random relative to a PA-degree $\mathbf{a}$. (Combining this idea with Harrington's forcing argument.)

Lemma (Simpson/Y 2011)

For any countable $(M, S) \models \mathrm{WWKL}_0$ *there is* $\hat{S} \supseteq S$ *satisfying*

1. $(M, \hat{S}) \models \mathrm{WKL}_0$*, and*
2. *for any* $A \in \hat{S}$ *there is* $z \in S$ *such that* z *is Martin-Löf random relative to* A*.*

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Lemma (Simpson/Y 2011)

For any countable $(M, S) \models \mathrm{WWKL}_0$ *there is* $\hat{S} \supseteq S$ *satisfying*

1. $(M, \hat{S}) \models \mathrm{WKL}_0$, *and*
2. *for any* $A \in \hat{S}$ *there is* $z \in S$ *such that* z *is Martin-Löf random relative to* A.

We will first formalize/modify the first and second steps.

- Every non-decreasing computable rationally presented function f_0 is differentiable at any Martin-Löf random real z.

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Model theoretic approach

By completeness theorem, we will show that any countable model of WWKL_0 satisfies the statement.

- Let $(M, S) \models \mathrm{WWKL}_0$ and $f \in S$.
- Take $\hat{S} \supseteq S$ by the second lemma.
- By Jordan decomposition theorem in WKL_0, we have non-decreasing $g, h \in \hat{S}$ such that $f = g - h$.
- By condition 2 of $\hat{S}$ take Martin-Löf random real relative to $g \oplus h$ z from S.
- By the first lemma, g and h are pseudo-differentiable at z, thus f is pseudo-differentiable at z. The latter holds in (M, S).

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Remark

The original proof of the following part actually uses RT^1.

> every non-decreasing rationally presented function f_0 is pseudo-differentiable at any Martin-Löf random real z.

To show this within RCA_0, we need a modified proof.

Remark

Rute showed that the existence of the derivative $f'(z)$ already requires ACA_0.

Questions

Question

Is there a reasonable way to interpret (or at least understand) results in computable analysis into reverse mathematics?

Some more technical questions.

Question

Is there some useful conservation between WWKL_0 and WKL_0 derived from the previous model-theoretic argument?

Question

What is the right strength of $\mathrm{WKL}(\textit{w-bd})$ (or $\mathrm{WKL}(\textit{ext-bd})$)?

Thank you!

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