STRONG MEASURE ZERO IN POLISH GROUPS

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The real line.

Definition. A set $A \subset \mathbb{R}$ is *strong measure zero* if for all $\langle \varepsilon_n \colon n \in \omega \rangle$ there is $\langle I_n \colon n \in \omega \rangle$ such that ength of I_n is $\leq \varepsilon_n$ and $A \subset \bigcup_n I_n$.

Conjecture. (Borel) All strong measure zero sets are countable.

Fact. (Laver) Borel conjecture is consistent.

Fact. (Galvin–Mycielski–Solovay) A set $A \subset \mathbb{R}$ is smz iff for every meager set $M \subset \mathbb{R}$, $A + M \neq \mathbb{R}$.

Generalizing to topological groups.

Definition. Let G be a second countable group. A set $A \subset G$ is *left strong measure zero* if for all $\langle U_n : n \in \omega \rangle$ there is $\langle g_n : n \in \omega \rangle$ such that $A \subset \bigcup_n g_n U_n$.

Fact. (Carlson) Borel conjecture for $\mathbb{R}$ implies the Borel conjecture for all Polish groups.

Fact. (Kysiak, Fremlin) If G is locally compact then a set $A \subset G$ is lsmz iff for every meager set $M \subset G$, $A \cdot M \neq G$.

Fact. (Hrušák–Zindulka) The characterization fails for $\mathbb{Z}^\omega$.

The motivating question.

Question. For which Polish groups G can we prove "$A \subset G$ is lsmz iff for all meager $M \subset G$, $A \cdot M \neq G$"?

1. the right-to-left implication always holds;

2. under Borel conjecture, the left-to-right implication holds as well.

Conjecture. Under CH, the characterization holds exactly for the locally compact groups.

Results.

Theorem. (CH) Suppose that G is a Polish group admitting bi-invariant metric, or G is a closed subgroup of S_∞. TFAE:

1. G is locally compact;

2. A set $A \subset G$ is lsmz iff for every meager set $M \subset G$, $A \cdot M \neq G$.

A technical tool.

Definition. A closed nowhere dense set $C \subset G$ is *bad* if for every $\langle U_n \colon n \in \omega \rangle$ there is $\langle g_n \colon n \in \omega \rangle$ such that for every $g \in G$, the set $C \cap g \cdot \bigcup_n g_n U_n$ is dense in C.

Fact. (Hrušák–Zindulka) (CH) If the group G contains a bad set then a transfinite recursion construction yields a lsmz set $A \subset G$ such that $A \cdot C^{-1} = G$.

A bad subset of S_∞.

Notation. If $n \in \omega$ and $t \in \omega^n$ is an injection then $[t] = \{g \in S_\infty \colon t \subset g\}$.

Construction. Let $\langle t_n \colon n \in \omega\rangle$ be finite injections such that $\mathrm{rng}(t_n) \subset \mathrm{rng}(t_{n+1})$ and $\bigcup_n [t_n] \subset S_\infty$ is dense. Then $C = S_\infty \setminus \bigcup_n [t_n]$ is bad.

Verification. Let $\langle i_n \colon n \in \omega\rangle$ be numbers. Pick g_n such that for every finite injection t there are m, n such that $t \subset g_m, g_n$ and $\mathrm{rng}(g_m \restriction i_m) \cap \mathrm{rng}(g_n \restriction i_n) \setminus \mathrm{rng}(t) = 0$. Then $\bigcup_n [g_n \restriction i_n]$ witnesses the badness of the set C.