Q

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Oaxaca, September 14, 2016

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Przymusiński (1980): $\exists$ Q-set $\Longrightarrow \exists$ Q-set of size $\aleph_1$ all of whose finite powers are Q.

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Theorem (Miller)

If $\exists$ Q-set of size $\aleph_2$ then there is a set of reals $X = \{x_\alpha : \alpha < \omega_2\}$ such that the set $\{(x_\alpha, x_\beta) : \alpha < \beta < \omega_2\}$ is a relative G_δ in X^2.

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Open Problem

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Open Problem (Miller)

CON ($\exists$ X such that X^2 is Q but X^3 is not).

Rough outline of proof

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no contradiction to Miller's result!!!

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$\forall$ $\mathbb{P}_{\kappa^+}$-name $\dot{A}$ for subset of κ $\exists \gamma < \kappa^+$ s.t. $\Vdash_\gamma \dot{A} = \dot{A}_\gamma$
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Corollary

$\Vdash_{\kappa^+} \{\dot{c}_\alpha : \alpha < \kappa\}$ *is a Q-set.*

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Lemma

Assume $p, q \in \mathbb{P}_\gamma$ are s.t.

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$\delta = 1$, limit step: obvious.

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$(\delta \geq 1)$ Prove for $\delta + 1$. Assume $(\alpha, n) \in r(\delta)$.
Wlog $(\alpha, n) \in p(\delta) \implies p{\restriction}\delta \Vdash \alpha \notin \dot{A}_\delta \implies r{\restriction}\delta \Vdash \alpha \notin \dot{A}_\delta$.

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Assume $(\alpha, n), (\sigma, n) \in r(\delta)$. Wlog $(\alpha, n) \in p(\delta)$ and $(\sigma, n) \in q(\delta)$.
So $\delta \in \mathrm{supp}(p) \cap \mathrm{supp}(q) \implies (\sigma, n) \in p(\delta)$
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($\delta \geq 1$) Prove for $\delta + 1$. Assume $(\alpha, n) \in r(\delta)$.
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Corollary

$\mathbb{P}_\gamma$ *is ccc,* $\gamma \leq \kappa^+$.

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This clearly implies $\Vdash (\dot{c}_\alpha, \dot{c}_\beta) \in \bigcap_n \dot{V}_n$. Done!

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Proof like compatibility lemma. Done!