The Ramsey degree of the pre-pseudoarc

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The pseudo-arc

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- continuum = compact and connected metric space;
- indecomposable = not a union of two proper subcontinua;
- chainable = each open cover is refined by an open cover $U_1, U_2, \ldots, U_n$ such that for i, j, $U_i \cap U_j \neq \emptyset$ if and only if $|j - i| \leq 1$

G – topological group

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Universal minimal flows

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The universal minimal flow of G is the unique minimal G-flow that has all other minimal G-flows as its homomorphic images.

Uspenskij's question

The following important question is open.

Question (Uspenskij, 2000)

Let P be the pseudo-arc and $H(P)$ be its homeomorphism group. Is the action $H(P) \curvearrowright P$ given by $(h, x) \to h(x)$ the universal minimal flow of $H(P)$?

Construction of the pseudo-arc, part 1

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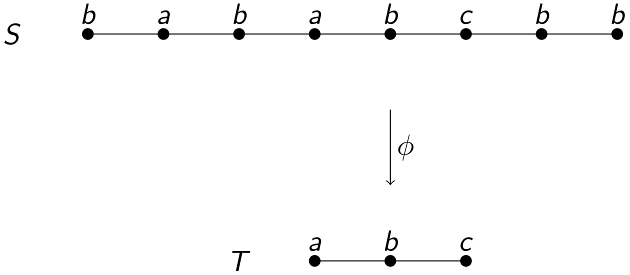
A continuous surjection $\phi\colon S \to T$ is an epimorphism iff

$$r^T(a, b) \iff \exists c, d \in S \left(\phi(c) = a, \phi(d) = b, \text{ and } r^S(c, d)\right).$$

An example of an epimorphism



Construction of the pseudo-arc, part 2

Theorem (Irwin-Solecki)

1. *The family $\mathcal{G}$ has the amalgamation property.*
2. *There is a unique $\mathbb{P} = (\mathbb{P}, r^{\mathbb{P}})$, where $\mathbb{P}$ is compact, separable, zero-dimensional, $r^{\mathbb{P}}$ is closed, which is projectively universal, projectively ultrahomogeneous, and continuous maps onto finite sets factor through epimorphisms onto finite structures.*
3. *The relation $r^{\mathbb{P}}$ is an equivalence relation such that each equivalence class has at most two elements.*

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Theorem (Irwin-Solecki)

$P = \mathbb{P}/r^{\mathbb{P}}$ *is the pseudo-arc.*

$\mathrm{Aut}(\mathbb{P})$ as a subgroup of $H(P)$

- Each automorphism $h \in \mathrm{Aut}(\mathbb{P})$ can be identified in a natural way with a homeomorphism $h^* \in H(P)$.

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- The topology on $\mathrm{Aut}(\mathbb{P})$ is finer than the compact-open topology on $H(P)$.
- The group $\mathrm{Aut}(\mathbb{P})$ is dense in $H(P)$.

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Ramsey degree

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Definition

Say that the Ramsey degree of A is $\leq t$ iff for any $A \leq B$ and any r there is $B \leq C$ such that for any coloring of $\binom{C}{A}$ into r colors there is $g \in \binom{C}{B}$ such that $\{f \circ g \colon f \in \binom{B}{A}\}$ is in $\leq t$ many colors.

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If there is no such t, say that the Ramsey degree is infinite.

Ramsey degree of $\mathcal{G}$

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The Ramsey degree of the 3-element structure in $\mathcal{G}$ is infinite.

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This result was also independently obtained by S. Solecki.

The universal minimal flow of $\mathrm{Aut}(\mathbb{F})$

Theorem (Zucker)

Infinite Ramsey degree for a structure in a Fraïssé family $\mathcal{F}$ $\iff$ *The universal minimal flow of* $\mathrm{Aut}(\mathbb{F})$*, the automorphism group of the projective Fraïssé limit of* $\mathcal{F}$*, is non-metrizable.*

Theorem (Zucker)

Infinite Ramsey degree for a structure in a Fraïssé family $\mathcal{F}$ $\iff$ The universal minimal flow of $\mathrm{Aut}(\mathbb{F})$, the automorphism group of the projective Fraïssé limit of $\mathcal{F}$, is non-metrizable.

Therefore we obtain:

Theorem

The universal minimal flow of $\mathrm{Aut}(\mathbb{F})$ is non-metrizable.

Sketch of the proof

- Take $B = 2(2^{2n+1} + 2)$ and $r = 2^n - 1$. Let $C > B$ be arbitrary.
- Want a coloring c such that for any $f \in \binom{C}{B}$ the set $\binom{B}{A} \circ f$ takes all 2^n colors.
- Let $z = \min\{i : h(c_i) = 1\}$.
  Let $Z = \min\{i > z : h(c_i) = 3\}$ and let $Z = 0$ if no such i.
- Define $\phi(h) = 0$ if $Z = 0$ and otherwise
  $\phi(h) = |\{z < i < Z : h(c_{i-1}) = 1 \ \& \ h(c_i) = 2\}|$.
- And we take the coloring $c(h) = \phi(h) \mod 2^n$.

Thank you!