REPORT ON CMO WORKSHOP: HARMONIC ANALYSIS, $\bar\partial$, AND CR GEOMETRY

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The CMO Workshop (15w5074) on Harmonic Analysis, $\bar\partial$, and CR Geometry was held on October 18-23, Oaxaca, Mexico. There were 34 participants from North America, Europe, South America, and Asia; among them were 5 Mexican mathematicians. During the workshop, 19 researchers gave 50-minute lecture on latest developments in the area. The talks covered a range of topics from classical several complex variables with the analysis of the Cauchy-Riemann and tangential Cauchy-Riemann operators and CR geometry. There were also several talks on topics from the intersection of harmonic analysis and several complex variables as well as pseudocomplex analysis. Below is a description of some of the activities.

1. L^2-theory of the $\bar\partial$ and $\bar\partial_b$ operators

L^2-theory of the $\bar\partial$-operator and the $\bar\partial$-Neumann Laplacian has played a key role in several complex variables and linear partial differential equations since the foundational work of Hörmander and Kohn in the 1960's. Recently, L^2-theory of the $\bar\partial$-operator has been applied to resolve several open problems in complex analysis and algebraic geometry. It also plays an important role in the recent work of X. Chen, S. Donaldson, and S. Sun on the Yau conjecture ([12]).

Professor Phillip Harrington from the University of Arkansas gave a talk on his recent work [26] on positivity of the Diederich-Fornæss exponent on pseudoconvex Lipschitz domain in $\mathbb{CP}^n$. The Diederich-Fornæss exponent is an important tool in L^2-theory of the $\bar\partial$-operator. In 1977, Diederich and Fornæss showed that for any bounded pseudoconvex domain Ω with C^2 boundary in a Stein manifold, there exist a positive constant η and a defining function r such that $\hat r$ =

$-(-r)^\eta$ is plurisubharmonic on Ω ([16]). This result of Diederich and Fornæss was generalized to bounded pseudoconvex domains with Lipschitz boundary by Harrington [24] (see [29, 15] for important earlier work of Kerzman-Rosay and Demailly). Quantitative relationship between the Diederich-Fornæss exponent and regularity of the $\bar\partial$-Neumann Laplacian on L^2-Sobolev spaces has been established by Kohn, Harrington and others (see [30, 25, 35]). In particular, this provides an effective approach to an earlier result of Boas and Straube [1] on global regularity of the $\bar\partial$-Neumann operator on a smooth bounded pseudoconvex domain with a defining function that is plurisubharmonic on the boundary. Berndtsson and Charpentier further showed that for a bounded pseudoconvex domain Ω with Lipschitz boundary in $\mathbb{C}^n$, the Bergman projection and the canonical solution operator for the $\bar\partial$-operator is bounded on L^2-Sobolev spaces $W^s(\Omega)$ for any s less than one half of the Diederich-Fornæss index ([3]; see also [7]). The Diederich-Fornæss index also plays a role in estimates of the pluri-complex Green function [5] and comparison of the Bergman and Szegö kernels [11].

For pseudoconvex domains in $\mathbb{CP}^n$, the situation is different, due to the lack of strongly plurisubharmonic exhaustion function. Takeuchi proved that the signed distance function of a (proper) pseudoconvex domain in $\mathbb{CP}^n$ with the Fubini-Study metric satisfies the strong Oka property ([37]) and Ohsawa and Sibony show that C^2-smoothly bounded pseudoconvex domain has a positive Diederich-Fornæss exponent ([34]). Whether or not Ohsawa-Sibony's result is also true for pseudoconvex domains in $\mathbb{CP}^n$ with Lipschitz boundary has been open since then until Harrington's solution.

In his lecture, Professor Andrew Raich discussed his recent joint work with Phillips Harrington ([27]) on the $\bar\partial$-problem on pseudoconvex unbounded domains in $\mathbb{C}^n$. He explained the difficulties in applying the established theory for bounded domains to unbounded ones and ways to overcome the problems. These included using weighted Sobolev spaces and finding good defining functions.

Professor Takeo Ohsawa from Nagoya University in Japan, who was recently awarded a Bergman Prize, gave a beautiful expository talk on background and open questions on L^2-extension theorems, an area where he has done fundamental work. In recent years, the L^2-extension theorems of Ohsawa and Takegoshi have been used to prove spectacular results in complex analysis and algebraic geometry by Berndtsson, Blocki, Demailly, Guan-Zhou, Paun, Siu and others. He explained some of these recent developments, in particular, the work of Blocki [6] and Guang-Zhou [22] on the Suita conjecture and those of Guang-Zhou [23], Berndtsson-Lempert [4] on the strong openness conjecture.

Professor Debraj Chakrabarti from Central Michigan State University discussed his recent joint work with Christine Laurent and Mei-Chi Shaw on the Dolbeault cohomology of the Chinese coin, annulus between the bidisc and a ball ([10]). He explained how this problem is related to W^1-estimate for the $\bar{\partial}$-operator on the bidisc.

2. Spectral theory of complex Laplacians

The $\bar{\partial}$-Neumann Laplacian is the archetype of an elliptic operator with non-coercive boundary conditions. Spectral properties of the $\bar{\partial}$-Neumann Laplacian are more sensitive to the underlying geometry than the usual Dirichlet or Neumann Laplacian. It follows from the fundamental work of Hörmander on L^2-estimates of the $\bar{\partial}$-operator and sheaf cohomology theory dated back to H. Cartan, J.-P. Serre and H. Laufer that one can determine pseudoconvexity of a bounded domain in $\mathbb{C}^n$ from positivity of the $\bar{\partial}$-Neumann Laplacian. Catlin showed that property (P), a potential theoretic property, is a sufficient condition for pure discreteness of the spectrum of the $\bar{\partial}$-Neumann Laplacian. For convex domains in $\mathbb{C}^n$ or pseudoconvex Hartogs domains in $\mathbb{C}^2$, property (P) is also a necessary condition ([19, 13]). It was discovered that pure discreteness of the spectrum on Hartogs domains in $\mathbb{C}^2$ is intimately related to diamagnetism and paramagnetism in semi-classical analysis of certain magnetic Schrödinger operators. How the spectrum of the $\bar{\partial}$-Neumann Laplacian interacts with the underlying geometric structure of the domain is a fascinating problem.

Professor Friedrich Haslinger from University of Vienna, Austria, spoke on his joint work [2] with Franz Berger on spectral properties of the weighted $\bar{\partial}$-Neumann problem on $\mathbb{C}^n$. He discussed a necessary condition on compactness of the weighted $\bar{\partial}$-Neumann operator on the space $L^2(\mathbb{C}^n, e^{-\phi})$, under some assumptions on ϕ and explained the difficulties in obtaining complete characterization of compactness in terms of properties of the weight function ϕ.

It is known that complex analytic disc on the boundary is an obstruction to discreteness of the spectrum of the $\bar{\partial}$-Neumann Laplacian (see [20]). In his talk, Professor Sonmez Sahutoglu from the University of Toledo discussed his joint work with Zeljko Cuckovic on a quantitative estimate of the failure of discreteness of the $\bar{\partial}$-Neumann Laplacian ([14]), particularly on worm domains.

3. Harmonic analysis and estimates of kernels

3.1. **Estimates of kernels.** In her talk, Professor Loredana Lanzani from Syracuse University discussed her recent joint work with Eli Stein

([32, 33]) on representations and density results for the classical Hardy spaces of boundary values of holomorphic functions, under minimal smoothness assumption on the boundary. She explained the difficulties in generalizing classical results from smooth domains to non-smooth domains and how she and Stein overcame these difficulties.

Professor Jeffrey McNeal from Ohio State University spoke on his joint work with Luke Edholm on the Bergman projection on generalized Hartogs triangles ([18]). The generalized Hartogs triangles are non-smooth domains between the classical Hartogs triangle and the product of the unit disc and punctured unit disc. He explained an interesting interplay between the defining functions of the generalized Hartogs triangles with L^p-mapping property of the Bergman projection.

3.2. **Harmonic analysis.** In the past thirty years, there has been a push to adapt harmonic analysis techniques and idea to the study of several complex variables and the naturally occurring integral operators in SCV. There is a paradigm carried over from classical harmonic analysis that there should be a relationship between the geometry and analysis on a domain. The SCV setting is more complicated than the real setting because elliptic regularity estimates for $\Box$ fail near the boundary of a domain and $\Box_b$ is never elliptic. Consequently, the classes of integral operators that mathematicians have developed are corresponding more complicated and able to capture subtle finite type behavior in $\mathbb{C}^2$ and certain special classes of domains in $\mathbb{C}^n$ (see the work of Nagel-Stein, Nagel-Stein-Wainger, Koenig, McNeal, etc.).

In his talk, Professor Brian Street from the University of Wisconsin-Madison gave an overview of theory of multiparameter singular integrals that he has developed. He gave examples to show their flexibility and their inclusion of all singular integral classes that are prevalent in real and complex analysis. In particular, they contain the classical Calderon-Zygmund classes, product theory classes, as well as flag kernels and non-isotropic smoothing operators.

Professor Po Lam Yung of The Chinese University of Hong Kong also spoke on a topic that relates to classical harmonic analysis and delicate endpoint results. It is well known that the Sobolev space $W^{1,n}(\mathbb{R}^n)$ does not embed into $L^\infty(\mathbb{R}^n)$ when $n \geq 2$. Recently, work by Bourgain, Brezis, van Schaftingen, Lanzani and Stein have found workarounds for the failure of this Sobolev embedding on $\mathbb{R}^n$. Professor Yung presented a version of this type of analysis in the context of certain symmetric spaces including complex hyperbolic spaces. Key is some analysis on the Heisenberg group.

4. CR Geometry

CR Geometry is an active developing area closely related to complex geometry and analysis of $\bar{\partial}$ equations. During recent decades fundamental results have been obtained, which in turn raised important open problems. Mapping problems and moduli spaces of CR structures have been subjects of recent research of many mathematicians including Baouendi, Ebenfelt, D'Angelo, Gong, Huang, Isaev, Kim, Lamel, Mir, Rothschild, Webster, Zaitsev, etc. An important direction is the local equivalence problem and normal forms of CR manifolds, that goes back to classical work of Cartan (1932), Tanaka (1962, 1967), and Chern–Moser (1974) on strictly pseudoconvex hypersurfaces.

Professor Ilya Kossovskiy (University of Vienna) gave a talk on an important aspect of the CR-equivalence problem. Identifying the 3-sphere among real hypersurfaces in 2-dimensional complex space is a problem which attracted a lot of attention of experts in CR-geometry since the work of Cartan and Chern-Moser. It is well known that the sphericity of a real hypersurface amounts to vanishing of its special CR-curvature. However, the latter is difficult to compute explicitly or identify geometrically. Kossovskiy discovered a simple unexpected geometric criterion of sphericity. It employs the Segre varieties of a real hypersurface. It turns out that a real-analytic hypersurface in 2-dimensional complex space is spherical if and only if its Segre family has the Desargues property, which is familiar from the Projective Geometry. This characterization is important because it does not assume any conditions on the hypersurface (e.g., existence of additional symmetries), nor it requires any special choice of coordinates.

An important fundamental question on the CR-equivalence problem is whether and when a formal solution to the problem gives an actual analytic or smooth solution. Professor Nordine Mir (Texas A&M University at Qatar) gave a presentation on this matter. According to Artin's famous approximation theorem of 1968, given any system of real-analytic equations, if there exists a formal solution to such a system at a given point, then there exists a real-analytic solution that is as close as we want in the Krull topology to the formal solution. One question that naturally thereafter arises is whether the conclusion of Artin's approximation theorem still holds if the system of equations is coupled with a specific PDE. In 1978, Milman investigated such a question when the PDE consists of the standard Cauchy-Riemann operator in $\mathbb{C}^n$: he showed that any formal solution of a system of real-analytic equations and of the standard CR equations in $\mathbb{C}^n$ can be approximated (in the Krull topology) by a sequence of convergent solutions of the system of analytic and CR equations. In his talk, Mir discussed recent results generalizing Milman's theorem when the standard Cauchy-Riemann operator in $\mathbb{C}^n$ is replaced by the tangential Cauchy-Riemann operator associated to a real-analytic CR manifold.

However, the straight answer to the main question whether a formal CR equivalence is analytic turns out to be negative. According to recent work by Kossovskiy and Shafikov [31], in every CR dimension and codimension, there is an example of a formal CR equivalence that is not analytic. Furthermore, for every positive integer k, there is an example of a C^k smooth CR equivalence that is not C^{k+1}. The cited work by Kossovskiy and Shafikov is a breakthrough in the CR equivalence problem.

There are fundamental difficult problems on deformations and embeddibility of abstract CR structures. The global problem for strongly pseudoconvex compact CR manifolds of hypersurface type of dimension at least 5 was solved by Boutet de Monvel (1975). Recently Chanillo, Chiu, and Yang (Duke, 2012) have made important progress. They prove the embeddibility of 3-dimensional CR manifolds with additional positivity assumption on Webster's curvature and CR Paneitz operator. Solutions of the local version of the problem were due to Kuranishi (1982), Akahori (1987), and Webster (1989) for strongly pseudoconvex CR manifolds of dimension at least 7.

Professor Gerardo Mendoza (Temple University) gave a talk on microlocal infinitesimal deformations of CR structures. Among the problems on local deformations and embeddibility of CR manifolds, the case of dimension 5 is particularly difficult. The equation expressing infinitesimal deformations of a strictly pseudoconvex almost CR structure of hypersurface type in dimension 5 consists locally of 4 first order equations in 6 unknowns. The study of this underdetermined system leads to a second order selfadjoint system whose characteristic variety is the span of the contact structure of the almost CR manifold and is hypoelliptic on one side of the characteristic variety. Mendoza presented new results on the microlocal structure of this system including the arguments leading to the expression of the system.

Professor Rafael Herrera from Centro de Investigation en Matematicas spoke on an interesting connection between CR structures on strictly pseudoconvex CR manifolds and twisted spin geometry (see [28] and references therein). He described how twisted spin geometry can be used to characterize CR structures of arbitrary codimension.

Professor Howard Jacobowitz from Rutgers University-Camden spoke on his on-going research on left invariant CR and pseudo-hermitian structures on S^3. He explained the classical work of Cartan and Webster, and discussed his results on classification of left invariant CR structures on S^3.

There has been intense interaction between CR geometry and work by many people in complex geometry (including N. Mok, X. Huang, Y. Yuan and many others) on local holomorphic isometries between Hermitian symmetric spaces: more specifically, rigidity questions and non-existence of common submanifolds. It is evident from Mok's work that products of balls play a special role in this area. Boundary considerations make CR geometry techniques effective for dealing with extension problems. In the talk by Professor Yuan Yuan, recent results by Yuan, Huang and Yuan, Xiao and Yuan were reported, related to maps into a product of balls, maps from a ball to a type IV domain, and non-existence of common submanifolds of a bounded symmetric domain and complex euclidean space.

5. Pseudocomplex Analysis

Professor Jingzhi Tie from the University of Georgia spoke about "Yaus Gradient Estimate and Liouville Theorem for Positive Pseudoharmonic Functions in a Complete Pseudohermitian manifold." Professor Tie first introduced the basic notion of pseudohermitian manifold and then derived the sub-gradient estimate for positive pseudoharmonic functions in a complete pseudohermitian $(2n + 1)$-manifold (M, J, θ) which satisfies the CR sub-Laplacian comparison property. The subgradient estimate served as the CR analogue of Yaus gradient estimate. He then obtained a Bishop-type sub-Laplacian comparison theorem in a class of complete noncompact pseudohermitian manifolds. That allowed him to establish the natural analogue of Liouville-type theorems for the sub-Laplacian in a complete pseudohermitian manifold of vanishing pseudohermitian torsion tensors and nonnegative pseudohermitian Ricci curvature tensors.

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