Women in Geometry 2

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1 Overview

There is currently an underrepresentation of women in the mathematics faculty of Ph.D. granting universities, a relative underrepresentation of women in Geometry/Topology and low visibility of women at Geometry conferences. For example, in the last fifteen years roughly 30% of all mathematics Ph.D.s in the United States have been awarded to women, with only 20% of these degrees awarded in Geometry/Topology to women. Despite this, the percentage of tenure-track and tenured female math faculty members at Ph.D. granting institutions in the U.S. is only 11%. (Statistics from Proceedings of the National Academy of Sciences publications as well as the Annual Survey of the Mathematical Sciences.)

The overarching goal of the Women in Geometry 2 (WIG 2) workshop was to increase the strength and visibility of the community of women geometers. The workshop hosted seven different research teams with anywhere from four to seven women each in the following areas: Conformal geometry, Geometric flows, Global Riemannian geometry, Mathematical general relativity, Minimal submanifolds, Spectral geometry and Symplectic geometry. Led by women established in these areas of geometric research, each team worked together on an open problem in their respective area. Before the program began team leaders provided participants with a synopsis of the research problem and necessary background reading materials. Once at BIRS, the majority of the time was spent in working groups pursuing collaborative research. Further background was developed through team seminars held during the first day, as needed. Seven plenary talks, one for each group, were scheduled during the first four mornings to provide attendees with new insights on current trends in geometry, to build a feeling of community, to promote discussion and collaboration within and between groups, as well as to inspire participants to consider expanding and broadening their own research programs. On the last day of the program each team reported back to the full workshop on progress made and future goals.

Other highlights of the program included a "Group Discussion" on Tuesday evening. The Group Discussion on Tuesday evening was an informal, but guided, conversation on the current status of women in mathematics with a view to finding ways to promote change. Topics included funding and job opportunities, improving the representation of women as speakers at conferences, how to lobby for funding for childcare to attend conferences, how to combat bias on award panels, and the mentoring of female graduate students as well as women in the early stages of their careers. Other topics included the composition of editorial boards, and the single blind peer review system in mathematics as opposed to the double blind peer review system in other scientific fields, such as computer science.

2 Collaboration with the Association for Women in Mathematics

WIG 2 is one of several similar conferences known collectively as Research Collaboration Conferences for Women (RCCW). In 2015 the Association for Women in Mathematics was awarded an NSF ADVANCE grant aimed at fostering the formation and growth of RCCWs. WIG 2 organizers collaborated with the AWM ADVANCE team by committing to maintain a conference website, to host a follow-up event at an AWM workshop, and to write a summary of the workshop for the AWM Newsletter. In exchange the AWM ADVANCE team provided some travel support for WIG 2 participants, as well as assessment of the workshop as part of a larger effort to understand the impact of RCCWs.

3 Presentation Highlights

There were seven plenary talks scheduled on the first four mornings of the workshop. A representative or representatives from each team gave a colloquium style lecture related to their team's research problem in one of these seven hour-long talks. The talks were accessible to all WIG II participants for at least the first 30-40 minutes. A goal of WIG was for participants to broaden their horizons a bit and get to know what are important problems in each other's areas. WIG II aimed to establish strong working relationships within teams. These lectures reached beyond this to lay the groundwork for possible cross-team collaboration.

Chikako Mese from Johns Hopkins University, representing the Minimal Submanifolds group, spoke on *Harmonic maps in dimension two*.

Alina Stancu from Concordia University, representing the Geometric Flows group, spoke on *Main tools and techniques in curvature flows - a user's guide*.

Regina Rotman from University of Toronto, representing the Global Riemannian Geometry group, spoke on *Periodic geodesics on Riemannian manifolds*.

Francesca De Marchis from Universita' di Roma "la Sapienza", representing the Conformal Geometry group, spoke on *Prescribing the Gaussian curvature on singular surfaces*

Mary Sandoval from Trinity College, representing the Spectral Geometry group, spoke on *Using heat invariants to distinguish orbifolds from manifolds*.

Emmy Murphy from Northwestern University, representing the Symplectic Geometry group, spoke on *Legendrian knots and Lagrangian cobordisms*.

Anna Sakovich from Uppsala Universitet, representing the Mathematical General Relativity group, spoke on *Understanding angular momentum in General Relativity*.

4 Scientific Progress Made

The majority of the time during the workshop was spent in working groups pursuing collaborative research. Here we summarize scientific progress made by each team. Each entry in the list contains: the team research area, the team members (with team leaders underlined), a description of the area, and a summary of the project and progress made during the workshop.

4.1 Conformal Geometry

Members:

Francesca De Marchis (Sapienza Univesità di Roma), Youngae Lee (Kyungpook National University), Yueh-Ju Lin (Wichita State University), Mariel Sáez (Pontificia Universidad Católica de Chile)

Area:

Given a Riemman Surface M with a Riemmanian metric g_0, a classical question in Conformal Geometry is whether a function K can be realized as the Gaussian curvature of a conformal metric $g = e^u g_0$. This problem was first proposed by Nirenberg and since then, several authors (such as Kazdan and Warner, Chang and Yang, Xu and Yang and many others) have found conditions on the function K that imply either existence or non-existence of such a metric g.
In particular on the standard sphere, for K rotationally symmetric it was proved in [1] that the problem can not be solved if K is monotone in the region where K is positive. On the contrary if K' changes sign in the set $\{K > 0\}$ and K is non-degenerate, then K can be prescribed as a Gaussian curvature of a conformal metric [6].

Troyanov [5] also considered the problem when conical singularities are present on the surface and we will focus in this case.

Summary:

During the workshop we focused on understanding conditions that would ensure existence (or non-existence) of metrics on $\mathbb{S}^2$, which are conformal to the standard round sphere, their prescribed Gaussian curvature is a given rotationally symmetric function K and have one conical singularity at a pole on the rotation axis. If $K = 1$ the problem is not solvable [5].
Following some arguments in [4] we easily proved that if K is sign-changing, negative at the singular point and monotone on the positive nodal region, the problem is not solvable. In the complementary cases there are some existence results [4, 3] but many situations are still open and we tried to use an ODE approach (as in [2]) to provide new sufficient conditions on K that guarantee solvability.

References

[1] W. X. Chen, C. Li, A necessary and sufficient condition for the Nirenberg problem. Comm. Pure Appl. Math. 48 (1995), no. 6, 657-667.

[2] K. S. Cheng, J. A. Smoller, Conformal metrics with prescribed Gaussian curvature on S2. Trans. Amer. Math. Soc. 336 (1993), no. 1, 219-251.

[3] T. D'Aprile, F. De Marchis, I. Ianni, Prescribed Gauss curvature problem on singular surfaces. Calc. Var. Partial Differential Equations 57 (2018), no. 4, Art. 99, 36 pp.

[4] F. De Marchis, R. Lopez-Soriano, Existence and non existence results for the singular Nirenberg problem. Calc. Var. Partial Differential Equations 55 (2016), no. 2, Art. 36, 35 pp.

[5] M. Troyanov, Prescribing curvature on compact surfaces with conical singularities. Trans. Amer. Math. Soc. 324 (1991), no. 2, 793-821.

[6] X. Xu, P.C. Yang, Remarks on prescribing Gauss curvature. Trans. Amer. Math. Soc. 336 (1993), no. 2, 831-840.

4.2 Geometric Flows

Members:

Theodora Bourni (University of Tennessee), Julie Clutterbuck (Monash University), Xuan Hien Nguyen (Iowa State University) Alina Stancu (Concordia University), Guofang Wei (UC Santa Barbara) Valentina Wheeler (via skype) (University of Wollongong)

Area:

Broadly speaking, geometric flows are gradient flows of functionals on manifolds which have certain geometric significance. Classical examples include Ricci flow, (inverse) mean curvature flow, harmonic map flow, and other variants. We focus on a flow by powers of curvature on the space of planar curves and on the heat flow with applications to spectral problems.

Summary:
We worked on the following two problems.

Fundamental gap in hyperbolic space

Here we considered the problem of finding a lower bound on the difference between the two lowest eigenvalues of the Laplace operator on a convex domain Ω. This difference is known as the fundamental gap and denoted by

$$\Gamma = \lambda_2 - \lambda_1.$$

In $\mathbb{R}^n$, the sharp bound is $3\pi^2/D$, where $D = \text{diameter}(\Omega)$. In $\mathbb{S}^n$ the same bound also holds. Both these bounds are found by comparison to a one-dimensional model problem. Our goal here was to extend this to domains in hyperbolic space.

In the above cases, a key tool is an improved log-concavity estimate on the first eigenfunction ϕ_1: that is, $\log \phi_1$ is not only concave, but at least as concave as the appropriate one-dimensional model solution.

We studied in depth a little-known result of Shih, in which an explicit domain was constructed with a non-concave first eigenfunction. This raises many questions, because in the flat case the quasiconcavity of such solutions (a weaker property than log-concavity) is extremely well studied, and for a very broad class of equations generally holds. We have generated many new questions regarding the solutions of elliptic equations on hyperbolic space.

Despite the log-concavity property not holding, we were able to find that the gap in Shih's example satisfies $\Gamma \geq 3\pi^2/D$. We expect that there exist other domains where the gap is smaller. We plan to write a short note on this to be published.

Curvature flow

Our main goal is to classify convex compact ancient solutions for the one-dimension flow by power of curvature (k^α, $1/2 < \alpha < 1$). Urbas showed the existence of translators in a slab for $1 \geq \alpha > 1/2$. When $\alpha \leq 1/2$, the translators are not in the slab but are entire graphs. For the curve shortening flow $\alpha = 1$, Daskalopoulos-Hamilton-Sesum did the classification, so we hope to extend their result to $\alpha \in (1/2, 1)$. One of the key steps in their construction is finding a so-called Lyapunov functional that is monotone under the flow and which will control the behavior as $t \to -\infty$. We currently have a good candidate for this monotone quantity. As $t \to -\infty$, the functional will give us information about the behavior (or the shape) of the possible ancient solutions. We conjecture that a convex compact ancient solution similar to the Angenent ovals, which can be seen as two translating solutions moving toward each other, exists for the flow by power of curvature $k^\alpha, 1/2 < \alpha < 1$.

We plan to continue to work on both problems via skype and look for opportunity to meet in the future. We have discussed applying to the SWiM program at MSRI, SQuaREs program at AIM, and/or the "Research in Pairs" at Oberwolfach.

4.3 Global Riemannian Geometry

Members:
Hannah Alpert (Ohio State U), Haydee Contreras Peruyero (UNAM Mexico City), Megan Kerr (Wellesley College), Regina Rotman (U Toronto), Catherine Searle (Wichita State U)

Area:
Global Riemannian Geometry generalizes the classical Euclidean, Spherical and Hyberbolic geometries. One of the major challenges is how local invariants such as curvature relate to global topological invariants such as fundamental group.

Summary:
We worked on two distinct projects and laid the groundwork for both. We plan to continue meeting via Skype and will be applying for funding to various places such as MSRI's SWiM program, Max Planck

Institute for Mathematics, and MFO Oberwolfach Research in Pairs, with a view to meeting again next summer to continue working on these two projects.

Positively curved manifolds of cohomogeneity-two

The overarching goal is to classify simply-connected, closed Riemannian manifolds of even dimension admitting a Lie group action with a G-invariant metric of positive curvature, for which the principal orbits have codimension 2.

By work of Fang, Grove, and Thorbergsson, it is known in this setting that if the G-action is polar, then the manifold is equivariantly diffeomorphic to a polar action on a compact rank one symmetric space. However, such actions have not been classified. Moreover, there are cohomogeneity two actions on standard spheres that are non-polar, by work of Straume.

While at CMO-BIRS, we worked out some restrictions due to positive curvature on the corresponding group diagrams, as well as some restrictions on the topological structure of such manifolds, which we plan to develop into a general structure theorem.

Existence of periodic geodesics in 3-manifolds with at least three ends

Thorbergsson proved that any complete non-compact surface with at least three ends has a periodic geodesic. Our goal is to generalize this result to dimension 3, that is, we would like to prove that the following conjecture holds:

Conjecture: *Suppose M^3 is a complete, non-compact manifold (with finite volume) and three or more ends. Then M has a periodic geodesic.*

While at CMO-BIRS, we began by studying the case where M^3 is a sphere with three points removed; each end is homeomorphic to $S^2 \times \mathbb{R}_+$ and were able to show that a periodic geodesic exists. From that starting point, we tackled more complicated topology, breaking the proof into these cases:

1. For some $k > 1$, $\pi_k(M) \neq 0$, and no ends are homeomorphic to $S^2 \times \mathbb{R}_+$;
2. M is a $K(\pi, 1)$ manifold, with no ends homeomorphic to $S^2 \times \mathbb{R}_+$; and finally,
3. at least one end of M^3 is is homeomorphic to $S^2 \times \mathbb{R}_+$.

We have completely proved Case 1 and we are optimistic that we will be able to prove the Conjecture in all cases.

4.4 Mathematical General Relativity

Members:

Maria Eugenia Gabach Clement (Universidad de Córdoba), Melanie Graf (Eberherd-Karls-Universität Tübingen), Anna Sakovich (Uppsala Universitet)

Area:

A century ago, Albert Einstein proposed his celebrated theory of General Relativity which explains gravity as curvature of spacetime, a geometric unification of space and time. Apart from its impact on cosmology and astrophysics, this theory has been an important source of interesting and challenging problems in geometry and partial differential equations. Methods from geometric analysis has played and continue to play a crucial role in the field. For example, minimal surface techniques are used in the celebrated proof of positive mass theorem by Schoen and Yau, while the proofs of Penrose inequality by Huisken and Illmanen, and Bray rely on the use of geometric flows, and conserved quantities such as center of mass can in many cases be defined in a coordinate-free way using geometric foliations (e.g. by surfaces of constant mean curvature), as it was first pointed out by Huisken and Yau.

Summary:

During the workshop our team focused on questions related to the notion of angular momentum for isolated systems, that is spacetimes modeling an object such as a black hole, a star or a galaxy in an otherwise empty universe. While the definitions of mass and linear momentum are well established in this setting, and recently there has been considerable progress in defining a suitable notion of center of mass, the notion of angular moment still remains rather obscure. On the one hand, we have the classical Hamiltonian definition of Beig and O'Murchadha given as a flux integral at infinity. This definition is very explicit but has some issues related to its well-definiteness and covariance with respect to asymptotic isometries. On the other hand, there is a very powerful quasilocal notion of angular momentum recently introduced by Chen, Wang, and Yau. The respective total notion of angular momentum obtained by taking the limit to spatial infinity does not have the above problems, but so far has only been computed under rather strong asymptotic assumptions.

Our main goal has been to study the relation between these two notions of angular momentum. In particular, we have initiated work proving that the two definitions agree if certain, rather general, conditions on the asymptotics at infinity are imposed. The main challenge is to solve the so-called optimal isometric embedding system – a system of geometrically motivated PDEs which is crucial for Chen, Wang and Yau's definition – in this setting, which we hope to achieve at least under a smallness hypothesis. We have also looked into a question regarding the limits of quasilocal conserved quantities along certain foliations by 2-surfaces that are commonly used in general relativity, and discussed a very basic question regarding what exactly is measured by these quantities. We have been able to formulate some conjectures and outlined a few directions in which we would like to proceed. We plan to apply for support so that we can meet again and continue to explore all these questions that we find very exciting.

4.5 Minimal submanifolds

Members:

Christine Breiner (Fordham University), Heather Macbeth (École normale supérieure), Chikako Mese (Johns Hopkins University), Raquel Perales (Universidad Nacional Autonoma de Mexico), Zahra Sinaei (University of Massachusetts at Amherst), Caroline VanBlargan (Johns Hopkins University)

Area:

Our group considered a question related to conformal harmonic maps (minimal surfaces) into metric spaces with upper curvature bounds. These maps are local minimizers for the energy functional. The conformality property implies that the maps are angle preserving. The curvature condition on the targets is analogous to a sectional curvature upper bound in the smooth setting. This bound is determined via comparison triangles on smooth model surfaces.

Summary:

Recall Teichmüller's Theorem: Let S be a surface and $[(\Sigma_1, h_1)], [(\Sigma_2, h_2)] \in \mathcal{T}(S)$ where Σ_1 and Σ_2 are compact Riemann surfaces. Then there exists a unique quasiconformal mapping $f_0 : \Sigma_1 \to \Sigma_2$ homotopic to $h_2 \circ h_1^{-1}$ with minimal dilatation. In 1954, Gerstenhaber and Rauch proposed proving the existence of the Teichmüller map via a variational principle. Let $\mathcal{M}(\Sigma_2)$ be a family of conformal metrics $g = \rho|dw|^2$ with area(g)=1 on a compact Riemann surface Σ_2 of genus ≥ 1. They conjectured that

$$\sup_{g\in\mathcal{M}} \inf_{f\in\mathcal{H}} {}^gE^f = \frac{1}{2}\left(K_* + \frac{1}{K_*}\right) \quad (1)$$

where $\mathcal{H}$ is the set of homeomorphisms homotopic to $h_2 \circ h_1^{-1}$ sufficiently nice such that the energy ${}^gE^f$ of f with respect to g makes sense and K_* is the minimal dilatation. Mese proved this result for compact surfaces. We consider the same problem for non-compact surfaces. We developed an understanding of the important techniques and ideas in the compact case and considered whether (and how) they might be modified in the non-compact case. There was much discussion on the types of non-compact ends that might appear. We also considered how the compactness theory for metrics must be refined in order to close the argument.

We plan to gather all six members of our group in New York during fall 2019, using funds from NSF-DMS 1750254 to support the travel of four visitors.

4.6 Spectral geometry

Members:

Katie Gittins (Université de Neuchâtel), Carolyn Gordon (Dartmouth College), Magda Khalile (Leibniz Universität Hannover), Ingrid Membrillo Solis (University of Southampton), Mary Sandoval (Trinity College), Elizabeth Stanhope (Lewis & Clark College)

Area:

In Spectral Geometry we ask: "What information about a geometric object is encoded in the eigenvalue spectrum of the Laplace operator of that object?" Informally this is stated "Can you hear the shape of a drum?" Varying the types of drums, as well as the spectra involved, has yielded a rich area of study.

Summary:

Our work was in the spectral geometry of Riemannian orbifolds. A Riemannian orbifold is a generalization of a Riemannian manifold in which the local structure is that of a quotient of a manifold under the action of a finite group of isometries. A simple example is the quotient of a disk by a finite cyclic group of rotations; the quotient is a cone and the cone point is an example of an orbifold singularity. Due to the local structure of orbifolds, the tools of spectral geometry, including the Laplace operator and its spectrum, can be carried over to the orbifold setting. The class of Riemannian orbifolds includes the class of Riemannian manifolds. A natural question is whether spectral data detects the presence of orbifold singularities.

A major tool for obtaining geometric information about a Riemannian manifold from spectral data is provided by the so-called "heat invariants", which form an infinite sequence of spectral invariants. In the more general setting of orbifolds, heat invariants have been studied for the Laplace operator acting on functions. Orbifolds are stratified spaces with the regular (i.e., non-singular) points forming an open dense stratum and the singular points forming lower-dimensional strata. The various strata each contribute to the heat invariants.

Our aims at WIG 2 were (i) to initiate the computation of heat invariants for the Hodge Laplacian on 1-forms for Riemannian orbifolds, and (ii) to determine whether the combination of information from the orbifold heat invariants for the Laplacian on functions together with the orbifold heat invariants for the Laplacian on 1-forms would be enough to distinguish a singular n-dimensional orbifold from a singular n-dimensional manifold for $n = 2, 3$. For $n = 2$, we note that some results were already obtained by Dryden, Gordon, Greenwald and Webb, but there were some remaining cases (which we have now dealt with).

Results:

We obtained a general expression for the lowest-level contribution of the singular strata to the heat invariants for the Laplacian on 1-forms and applied this result to obtain an affirmative answer to the question raised in (ii) above.

Next steps and future plans:

Our group has sketched a plan to complete a manuscript establishing these results with a submission deadline in the summer of 2020. We also hope to meet again during the summer of 2020.

4.7 Symplectic geometry

Members:

Orsola Capovilla-Searle (Duke University), Maia Fraser (University of Ottawa), Maÿlis Limouzineau (Universität zu Köln), Emmy Murphy (Northwestern University), Yu Pan (MIT), Lisa Traynor (Bryn Mawr)

Area:

Symplectic and contact structures are closely related geometric structures that can be put on even (respectively, odd) dimensional smooth manifolds. By Darboux's theorem these structures are locally unique so that all invariants are global in nature. A key object of study in contact (respectively symplectic) manifolds are so-called Legendrian (respectively Lagrangian) submanifolds. In particular, Lagrangian cobordisms between Legendrian submanifolds are an especially rich source of information that can in turn help in understanding flexibility/rigidity phenomena in contact manifolds.

Summary:

The symplectic geometry group focused on problems related to exact Lagrangian cobordisms between Legendrian knots. Before the workshop began, we formulated three problems for possible collaboration. The majority of the time was spent on a problem exploring the relationship between embedded and immersed Lagrangian fillings of Legendrian knots. Specifically, every immersed Lagrangian filling gives rise to an embedded filling of higher genus, by surgering the double points. The converse however is unknown: whether every embedded filling can be presented as surgery on an immersed disk.

We found a candidate for a Legendrian knot which does have an embedded genus 1 filling but does not have a filling by a Lagrangian disk with one double point. We were able to prove that there does not exist a "decomposable" immersed filling of this type. We planned some strategies to prove the result without the decomposable hypothesis, which mostly involves generalizing a result about injectivity of augmentations from the case of cobordisms between Legendrian knots, to Legendrian links. This proof is nearing its completion and will enter the writing stage in the near future.

We also spent a significant amount of time on another problem, exploring how decomposable Lagrangian cobordisms affect destabilizability of Legendrians. A natural conjecture is that non-destabilizable Legendrians are sent to non-stabilizable Legendrians under decomposable cobordisms. The proof strategy we explored involves possible transverse intersections between standard surfaces in contact 3-space, how those intersections interacts with the characteristic foliations on the surfaces, and methods of surgering the intersections away. This project still has a significant amount of work necessary to complete, but the lemmas we were able to prove during the week are promising. We intend to continue this project in the coming year.

5 Outcome of the Meeting

Each of the research teams made significant progress on their research problems during the workshop. The teams plan to continue their collaborations. Some teams are having skype meetings regularly and in order to meet again are applying to research in teams programs, such as SWiM at MSRI, SQuaRE at AIM, Research in Paris, the Max Planck Institute for Mathematics in Bonn, and the MFO in Oberwolfach.

The feedback on the workshop that we have received from participants is quite positive and resulting publications are expected. A survey administered by the AWM ADVANCE team immediately after WIG2 assessed participants' views on the effectiveness of the workshop. The response rate to the survey was 77%. The AWM ADVANCE team reported the results of the survey to us, and from that report we excerpt the following: "Based on this report, the WIG RCCW appears to have been very successful. Enthusiasm for the RCCW is high, as are participants' desires to participate again in the future. The workshop appeared to assemble a motivated and productive group of women..."

Participants benefited on the individual level by building background knowledge on a new problem, by strengthening and broadening their research programs, and, in some cases, by being provided with a re-entry point after being sidetracked by any or all of family duties, high service loads or high teaching loads. By building teams that included women at all career stages, from advanced graduate students and recent Ph.D.s to associate professors seeking to invigorate their research programs to senior researchers, the workshop formed mentoring and collaborative networks that will strengthen the careers of all participants. All attending gained an overview of seven exciting areas of current research in geometry, and all contributed to progress in their own area.

The community of women geometers was strengthened by the supportive research community, mentorship of women just beginning or at the middle of their research careers, and the new collaborative links forged between women geometers working within and between their respective areas of specialization at WIG2. It is important to note that the areas of geometry featured in the WIG2 program are strongly interrelated, so the potential for cross-area collaboration is high. The visibility of the community of women geometers was increased by highlighting the work of established female leaders in geometry, by bringing attention to the work of outstanding new women geometers, and, very simply, by having this many women together to do geometry research.

Finally, it bears mentioning that five research articles resulted from the first Women in Geometry workshop held at the Banff International Research Station. Based on the preliminary information we have received from the participants of this second Women in Geometry workshop, we believe that at least as many, if not more, may be expected from WIG2.